Math 2050B, HW 5. Due: 30 Nov 2024, before 11:59 pm

(1) Show that $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \frac{1}{x^2+1}$$
is uniformly continuous by using ε-δ terminology.

(2) Suppose $f : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $f|_{[a,+\infty)}$ is uniformly continuous for some $a > 0$. Show that f is uniform continuous.

(3) If $f, g : \mathbb{R} \to \mathbb{R}$ are two uniform continuous function, show that $f \circ g$ is also uniform continuous.

(4) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that
$$\lim_{x\to+\infty} f(x) = L_1, \lim_{x\to-\infty} f(x) = L_2$$
for some L_i. If $f(0) > \max\{L_1, L_2\}$, show that there exists $\bar{x} \in \mathbb{R}$ such that $f(\bar{x}) \geq f(x)$ for all $x \in \mathbb{R}$.

(5) Let A be a compact set in $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is a real valued function such that for any $\varepsilon > 0$, there is a polynomial g_ε such that $\sup_A |f(x) - g_\varepsilon(x)| < \varepsilon$. Show that f is uniformly continuous.

(6) Suppose $f : (0, 1] \to \mathbb{R}$ is a bounded continuous function. Show that the function given by $g(x) = xf(x)$ is uniformly continuous on $(0, 1)$.