THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Suggested Solutions for Quiz 2

1. Using the $\varepsilon - \delta$ terminology, show
$$\lim_{x\to 0}\frac{\sqrt{x+1}}{x+1} = 1.$$

Solution. Let $\varepsilon > 0$ be given. We have that
$$\begin{aligned}\left|\frac{\sqrt{x+1}}{x+1} - 1\right| &= \left|\frac{\sqrt{x+1}-(x+1)}{x+1}\right| = \left|\frac{\sqrt{x+1}-(x+1)}{x+1}\cdot\frac{\sqrt{x+1}+(x+1)}{\sqrt{x+1}+(x+1)}\right| \\ &= \left|\frac{-x^2-2x}{(x+1)(\sqrt{x+1}+(x+1))}\right| \le |x|\left|\frac{-(x+2)}{(x+1)(\sqrt{x+1}+(x+1))}\right|.\end{aligned}$$
When $-\frac{1}{2} < x < \frac{1}{2}$, after some algebra, one can show that
$$\left|\frac{-(x+2)}{(x+1)(\sqrt{x+1}+(x+1))}\right| < 6(\sqrt{2}-1)$$
and so taking $\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{6(\sqrt{2}-1)}\right\}$, we see that whenever $|x| < \delta$, $\left|\frac{\sqrt{x+1}}{x+1} - 1\right| <$ ε as required. ◀

2. Let $f : (0, +\infty) \to \mathbb{R}$ be a function given by
$$f(x) = \frac{1}{x^2+a^2}.$$
Show that f is uniformly continuous if $a > 0$. What if $a = 0$? Justify your answer.

Solution. Let $\varepsilon > 0$ be given. Let $x, y \in (0, +\infty)$. Note that if $x < a$, then we see that $\frac{x}{x^2+a^2} < \frac{a}{x^2+a^2} < \frac{1}{a}$, and if $a < x$, then we see that $\frac{x}{x^2+a^2} < \frac{x}{x^2} < \frac{1}{x} < \frac{1}{a}$. So either way, $\frac{x}{x^2+a^2} < \frac{1}{a}$ for $x \in (0, +\infty)$. We have
$$\begin{aligned}\left|\frac{1}{x^2+a^2} - \frac{1}{y^2+a^2}\right| &= \left|\frac{y^2-x^2}{(x^2+a^2)(y^2+a^2)}\right| \\ &\le |y-x|\left(\left|\frac{y}{(x^2+a^2)(y^2+a^2)}\right| + \left|\frac{x}{(x^2+a^2)(y^2+a^2)}\right|\right) \\ &\le |y-x|\cdot\frac{2}{a^3}.\end{aligned}$$
So setting $\delta := \frac{\varepsilon a^3}{2}$ yields the desired result.

When $a = 0$, the function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, +\infty)$. ◀

3. Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < 0 < f(b)$. Let $S := \{c \in [a, b] : f(c) < 0\}$.

   (a) Show that $\gamma = \sup S$ exists.

   (b) Show that $f(\gamma) = 0$.

   Solution. (a) Since we have that $f(a) < 0$, we know that $a \in S$ and S is non-empty. Since $f(b) > 0$, we know that S is bounded above by b. Hence, $\gamma = \sup S$ exists by the completeness of $\mathbb{R}$.

   (b) Since f is continuous, by the intermediate value theorem, there is a $x \in [a, b]$ such that $f(x) = 0$. It is clear that x is an upper bound of S and so we have that $\gamma \leq x$. Suppose $\gamma < x$ and suppose $f(\gamma) < 0$. Then by the intermediate value theorem, we have that there is a $\gamma < z_1 < x$ such that $f(\gamma) < f(z_1) < f(x) = 0$. But this means $z_1 \in S$ and contradicts the fact that γ is an upper bound of S. So either $\gamma = x$, in which case we would have $f(\gamma) = f(x) = 0$, or $f(\gamma) \geq 0$. If $f(\gamma) > 0$, then a similar argument shows that we can find a $z_2 < \gamma$ such that $0 < f(z_x) < f(\gamma)$ which contradicts the fact that γ is the least upper bound.

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