THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 9

Date: 21 November, 2024

already did these.

1. Use either the $\varepsilon - \delta$ definition of limit or the Sequential Criterion for limits, to establish the following:

   (a) $\lim_{x\to 0} \frac{x^2}{|x|} = 0$

   (b) $\lim_{x\to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$

   (c) $\lim_{x\to 0} \frac{1}{\sqrt{x}}$, $(x > 0)$ does not exist.

2. Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous of $A = [1, \infty)$ but not uniformly continuous on $(0, \infty)$.

3. (Exercises 5.2.5-5.2.6 of [BS11])

   (a) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim_{x\to c} f = b$ and that g is continuous at b. Show that $\lim_{x\to c} g \circ f = g(b)$.

   (b) Does the conclusion hold if g is not continuous at b? Give an example showing otherwise.

4. Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $f([0, 1]) \subset \mathbb{Q}$. Show that f is a constant function.

1. Use either the $\varepsilon - \delta$ definition of limit or the Sequential Criterion for limits, to establish the following:

   (a) $\displaystyle\lim_{x\to 0} \frac{x^2}{|x|} = 0$

   (b) $\displaystyle\lim_{x\to 1} \frac{x^2 - x + 1}{x+1} = \frac{1}{2}$

   (c) $\displaystyle\lim_{x\to 0} \frac{1}{\sqrt{x}}$, $(x > 0)$ does not exist.

2. Show that the function $f(x) = \dfrac{1}{x}$ is uniformly continuous of $A = [1, \infty)$ but not uniformly continuous on $(0, \infty)$.

Pf: Let $\varepsilon > 0$ be given and let $x, y \in [1, \infty)$. Then $\frac{1}{x}, \frac{1}{y} \leq 1$.

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| \leq |y-x| \left|\frac{1}{x}\right| \left|\frac{1}{y}\right| \leq |y-x|.$$

So taking $\delta = \varepsilon$. if $x, y \in A$ with $|x-y| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{y}\right| \leq |y-x| < \varepsilon.$$

Showing f is not uniformly continuous on $(0, +\infty)$. WTS $\exists \varepsilon_0 > 0$ s.t. for all $\delta > 0$, we can find $x_0, y_0 \in (0, +\infty)$ s.t. $|x_0 - y_0| < \delta$ but

$$|f(x_0) - f(y_0)| \geq \varepsilon_0.$$

Let's take $\varepsilon_0 = 1$. Then for all $\delta > 0$, by AP., there is a $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$. So taking $x_0 = \frac{1}{N}$, $y_0 = \frac{1}{2N}$.

$$|x_0 - y_0| = \left|\frac{1}{N} - \frac{1}{2N}\right| = \frac{1}{N} < \delta$$

but $$|f(x_0) - f(y_0)| = \left|\frac{1}{1/N} - \frac{1}{1/2N}\right| = N \geq 1 = \varepsilon_0.$$

3. (Exercises 5.2.5-5.2.6 of [BS11])

   (a) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that $\lim_{x\to c} f = b$ and that g is continuous at b. Show that $\lim_{x\to c} g \circ f = g(b)$.

   (b) Does the conclusion hold if g is not continuous at b? Give an example showing otherwise.

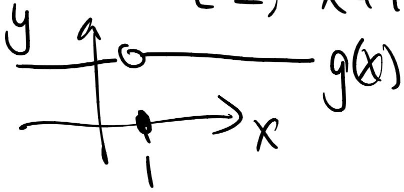
Pf: a). Let $\varepsilon > 0$. Since g is cts at b, $\exists\ \delta > 0$ s.t. for $x \in \mathbb{R}$ with $|x-b| < \delta$, then $|g(x)-g(b)| < \varepsilon$.

Moreover since $\lim_{x\to c} f = b$ for $\varepsilon' = \delta$, we can find $\delta' > 0$ such that if $0 < |x-c| < \delta'$, then $|f(x) - b| < \varepsilon' = \delta$.

So replacing x with $f(x)$ above, we obtain for $0 < |x-c| < \delta'$,

$$|f(x) - b| < \delta \Rightarrow |g(f(x)) - g(b)| < \varepsilon.$$

b) $g(x) = \begin{cases} 0, & x = 1 \\ 2, & x \neq 1. \end{cases}$ $\quad f(x) = x+1.$



Then $\lim_{x\to 0} f = 1$.

But we'll show that $\lim_{x\to 0} g\circ f \neq (g\circ f)(0)$.

$(g\circ f)(0) = g(f(0)) = g(1) = 0.$

Let (x_n) be any sequence in $\mathbb{R}$ s.t. $x_n \neq 0$ for all $n \in \mathbb{N}$ and $x_n \to 0$

Then $(g\circ f)(x_n) = g(f(x_n)) = g(x_n + 1) \equiv 2.$

So $\lim_{x\to 0} (g\circ f)(x) = 2 \neq 0 = (g\circ f)(0).$

4. Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function such that $f([0,1]) \subset \mathbb{Q}$. Show that f is a constant function.

Pf: Suppose f is not constant. Then there exist $x_1, x_2 \in [0,1]$ s.t. $\underbrace{f(x_1)}_{\in \mathbb{Q}} < \underbrace{f(x_2)}_{\in \mathbb{Q}}$. By density of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$, there is an $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ s.t. $f(x_1) < \alpha < f(x_2)$.

Then IVT tells us that there is a $c \in [0,1]$ s.t.

$f(c) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$ a contradiction.

//.