THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Revision Class

Date: 20 November, 2024

Announcement:

Quiz 2 TOMORROW (21/11) 1530 - 1615.

Midterms have been graded but may have some adjustment by Prof. Lee. So in the end I will not announce the statistics today.

1. Recall the following definitions and theorems:

   (a) Cluster point
   (b) Limit of a function
   (c) Sequential criterion for limits of functions
   (d) Divergence criteria:
   (e) Limits at infinity
   (f) Continuity
   (g) Sequential criterion for continuity
   (h) Discontinuity criterion
   (i) Maximum-minimum theorem for continuous functions
   (j) Intermediate value theorem
   (k) Preservation of intervals theorem
   (l) Uniformly continuity
   (m) Nonuniform continuity criteria
   (n) Uniform continuity theorem for continuous functions on closed bounded intervals
   (o) Continuous extension theorem

a) Cluster point: Let $A \subseteq \mathbb{R}$. Then $c \in \mathbb{R}$ is a cluster point of A if for every $\delta > 0$, there is a $x \in A$ s.t. $|x - c| < \delta$.

b) Limit of a function: $f: A \subseteq \mathbb{R} \to \mathbb{R}$ and let c be a cluster pt. of A. Then a real number L is the limit of f at c, written $\lim_{x \to c} f(x) = L$ if for all $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in A$ and $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$.

c) Sequential criterion for limit of a function: $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ a cluster point of A. TFAE:

1) $\lim_{x \to c} f(x) = L$

2) For every sequence $(x_n) \subseteq A$ s.t. $x_n \to c$ and $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

d) Divergence Criterion: Let $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ a cluster point of A.

1) If $L \in \mathbb{R}$ then f does not have a limit L at c iff there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$ but the sequence $(f(x_n))$ does not converge to L.

2) f does not have a limit at c iff there exists a sequence (x_n) in A with $x_n \neq c$ and $x_n \to c$ but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.

Ex: Write down the $\varepsilon-\delta$ version of this def'n.

e) Limit at infinity: $\lim_{x \to +\infty} f(x) = L$ if given any $\varepsilon > 0$, there is a $K(\varepsilon) > 0$ s.t. if $x \geq K$, $|f(x) - L| < \varepsilon$.

Ex: Write down def'n of $\lim_{x \to -\infty} f(x) = L$.

i) Max-min Thm for cts fn's: let $I := [a,b]$ be a closed bounded interval. and let $f: I \to \mathbb{R}$ be continuous on I. Then f attains maximum and minimum on I.

j) IVT: let I be an interval and $f: I \to \mathbb{R}$ be continuous. If $a, b \in I$ and if $k \in \mathbb{R}$ which satisfies $f(a) < k < f(b)$, then there exists a $c \in I$ s.t. $f(c) = k$.

n) Unf. Cty thm for cts f'n's on closed bdd intervals: If $f: I \to \mathbb{R}$ is cts where I is a closed bounded interval, then f is uniformly cts on I.

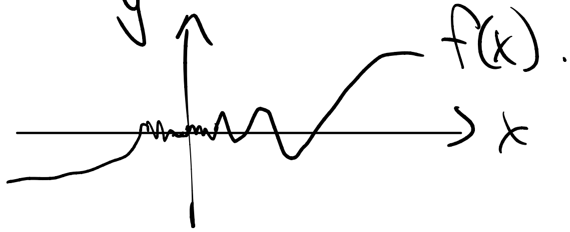
o) Continuous extension theorem: f is uniformly continuous on (a,b) iff it can be defined at the end points a, b s.t. the extended function is continuous on $[a,b]$.

$f: (a,b) \to \mathbb{R}$ unf. cts on (a,b) iff there exists $g: [a,b] \to \mathbb{R}$ s.t. $g|_{(a,b)} = f$ and g is continuous.

9. Show that the function $f : (0,1) \to \mathbb{R}$ given by

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

is uniformly continuous.



Pf: Method 1: Using continuous extension theorem.

f is continuous on $(0,1)$, and easy to show $\lim_{x\to 1} f(x) = \sin(1)$.

Note that $|f(x)| \leq |x|$ for all $x \in (0,1)$.

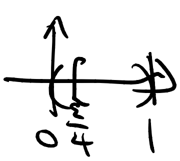
So by Squeeze Thm, $\lim_{x\to 0} |f(x)| \leq \lim_{x\to 0} |x| = 0$.

So $\lim_{x\to 0} f(x) = 0$

So define $g(x) = \begin{cases} 0 & , x = 0 \\ x\sin(\frac{1}{x}) & , x \in (0,1) \\ \sin(1) & , x = 1 \end{cases}$

and by construction g is continuous on $[0,1]$. Hence by cts extension theorem, f is unif. cts. on $(0,1)$.

Method 2: Using ε-δ. Let $\varepsilon > 0$. Note that f is cts on $[\frac{\varepsilon}{4}, 1]$. So f is unif. cts. on $[\frac{\varepsilon}{4}, 1]$. So there is a $\delta_0 > 0$



s.t. if $x, y \in [\frac{\varepsilon}{4}, 1]$ with $|x-y| < \delta_0$, we have $|f(x) - f(y)| < \varepsilon$.

Set $\delta = \min\{\delta_0, \frac{\varepsilon}{4}\}$.

Then case 1: x or $y \in (0, \frac{\varepsilon}{4})$ and $|x-y| < \delta$.

WLOG suppose $x \in (0, \frac{\varepsilon}{4})$, then by triangle inequality, we have

$|y| = |y+x-x| \leq |y-x| + |x| < \delta + \frac{\varepsilon}{4} < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$.

So $x, y \in (0, \frac{\varepsilon}{2})$. Then using the fact that $-x < f(x) < x$, we have that $-\frac{\varepsilon}{2} < f(x) < \frac{\varepsilon}{2}$

$-\frac{\varepsilon}{2} < f(y) < \frac{\varepsilon}{2}$.

So $|f(x) - f(y)| < \varepsilon$.

Case 2: Suppose both $x, y \notin (0, \frac{\varepsilon}{4})$. Then $x, y \in [\frac{\varepsilon}{4}, 1) \subseteq [\frac{\varepsilon}{4}, 1]$.
and by above, using δ_0, we have that $|f(x) - f(y)| < \varepsilon$.
So f is unif. cts. on $(0,1)$.

//.

10. Show that $f : (a, +\infty) \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x^2}$ is uniformly continuous if $a > 0$ but fails to be if $a = 0$.

Pf: Suppose $a>0$. Let $\varepsilon>0$. Then if $x, y \in (a, +\infty)$, we have that $x, y > a > 0 \Rightarrow \frac{1}{x}, \frac{1}{y} \leq \frac{1}{a}$.

Then

$$|f(x)-f(y)| = \left|\frac{1}{x^2}-\frac{1}{y^2}\right| = \left|\frac{y^2-x^2}{x^2y^2}\right| \leq |x-y|\left|\frac{1}{xy^2}+\frac{1}{yx^2}\right|$$

$$\leq \frac{2}{a^3}|x-y|$$

So taking $\delta = \frac{a^3\varepsilon}{2}$, we have that if $|x-y|<\delta$, then $|f(x)-f(y)| < \frac{2}{a^3}\cdot\frac{a^3\varepsilon}{2} = \varepsilon$.

Now suppose $a=0$. Use sequential criterion: WTS $\exists \varepsilon_0>0$ s.t. there are sequences (x_n), (y_n) with $\lim_{n\to\infty} x_n - y_n = 0$ but $|f(x_n)-f(y_n)| \geq \varepsilon_0$.

Take $\varepsilon_0 = 1$. $x_n = \frac{1}{\sqrt{n}}$, $y_n = \frac{1}{\sqrt{n+1}}$. Clearly $\lim_{n\to\infty} y_n - x_n = 0$.

but $|f(x_n)-f(y_n)| = \left|\frac{1}{(\frac{1}{\sqrt{n}})^2} - \frac{1}{(\frac{1}{\sqrt{n+1}})^2}\right| = |n-n+1| = 1 = \varepsilon_0$.

ε-δ version of showing f is not unif. cts at c: $\exists \varepsilon_0>0$ s.t. for all $\delta>0$, there are $x, y \in A$ s.t. $|x-y|<\delta$ but $|f(x)-f(y)| \geq \varepsilon_0$.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to +\infty} f(x) = L$, $\lim_{x \to -\infty} f(x) = L'$. Show that f is uniformly continuous.

"Generalization of unif. cty thm for cts fn's on I"

Pf: Let $\varepsilon > 0$. Then by assumption on limits at $\pm\infty$, $\exists M, M' > 0$ s.t. if

$x \geq M$, $|f(x) - L| < \frac{\varepsilon}{4}$.

$x \leq -M'$, $|f(x) - L'| < \frac{\varepsilon}{4}$.

We also have that on $[-M', M]$, f is unif. cts, so $\exists \delta_0 > 0$ s.t. if $x, y \in [-M', M]$, and $|x - y| < \delta_0$, $|f(x) - f(y)| < \frac{\varepsilon}{4}$.

So let $x, y \in \mathbb{R}$ and $|x - y| < \delta_0$. Then

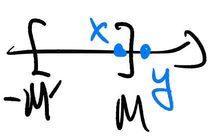
Case 1: if both $x, y \geq M$, then

$$|f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \frac{\varepsilon}{2}.$$

Case 2: if both $x, y \leq -M$, then

$$|f(x) - f(y)| \leq |f(x) - L'| + |L' - f(y)| < \frac{\varepsilon}{2}.$$

Case 3: Suppose $x \in [-M', M]$ and $y \in (M, +\infty)$ and $|x - y| < \delta_0$



Then by triangle inequality, $x, y \in (M - \delta_0, M + \delta_0)$ and so

$$|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - L| + |L - f(y)|$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4} < \varepsilon.$$

Other cases are similar

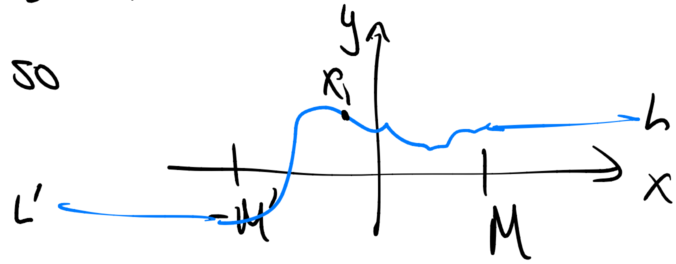
12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L'$, show that $f(\mathbb{R})$ is a bounded set. If there exists x_1, x_2 such that $f(x_1) > \max\{L, L'\}$ and $f(x_2) < \min\{L, L'\}$, show that $f(\mathbb{R})$ is a closed and bounded set and moreover, its extremums are attained.

Pf: Boundedness: Take $\varepsilon = 1$. Then $\exists M, M' > 0$ s.t. if $x \geq M$,
$|f(x) - L| < 1$
$\Rightarrow |f(x)| < 1 + L$.

and also if $x \leq -M'$, then $|f(x) - L'| < 1$
$\Rightarrow |f(x)| < 1 + L'$

So on $(-\infty, -M'] \cup [M, +\infty)$, f is bounded.

On $[-M', M]$, f is continuous, so bounded on $[-M', M]$.



So f is bounded on $\mathbb{R}$.

$f(\mathbb{R})$ closed: it's enough to show f attains max and min on $\mathbb{R}$. because if $\exists c_1, c_2 \in \mathbb{R}$ s.t. $f(c_1)$ is max, $f(c_2)$ is min, then $f(\mathbb{R}) = [f(c_2), f(c_1)]$ by IVT. Check this!

WTS f attains max in $\mathbb{R}$. Since $f(x_1) > \max\{L, L'\}$, then $f(x_1) - L > 0$, $f(x_1) - L' > 0$. So take $\varepsilon_1 = f(x_1) - L$.

Then by $\lim_{x\to+\infty} f(x) = L$, $\exists N > 0$ s.t. if $x \geq N$,

$$|f(x) - L| < \varepsilon = f(x_1) - L \Rightarrow f(x) < f(x_1) \quad \text{for } x \geq N.$$

Similarly, taking $\varepsilon_2 = f(x_1) - L'$, $\exists N' > 0$ s.t. if $x \leq -N'$,
$f(x) < f(x_1)$.

So f attains max at x_1 for $x \in (-\infty, -N'] \cup [N, +\infty)$.

On $[-N, N]$, since f is continuous, f attains max on $[-N, N]$ at some $c \in [-N, N]$. So $\max_{\mathbb{R}} f = \max\{f(x_1), f(c)\}$.

Similarly can show f attains min on $\mathbb{R}$. Do this!

13. $f : [a,b] \to \mathbb{R}$ is continuous such that for all $x \in [a,b]$, there is $y \in [a,b]$ such that $|f(y)| \le \frac{1}{2}|f(x)|$. Show that $f(c) = 0$ for some $c \in [a,b]$.

Pf: Take $x_1 \in [a,b]$. Then $\exists\, x_2 \in [a,b]$ s.t. $|f(x_2)| \le \frac{1}{2}|f(x_1)|$

$\exists\, x_3 \in [a,b]$ s.t. $|f(x_3)| \le \frac{1}{2}|f(x_2)| \le \frac{1}{2^2}|f(x_1)|$

$\vdots$

$\exists$ sequence $(x_n) \subseteq [a,b]$ s.t. $|f(x_n)| \le \frac{1}{2^n}|f(x_1)|$.

"take limit as $n \to \infty$" but x_n may not converge!

x_n is a bounded sequence ($a \le x_n \le b$ for all $n \in \mathbb{N}$)

So by BWT there is a subsequence $x_{n_k} \to c \in [a,b]$.

Then since f, $|\cdot|$ are continuous

$$|f(c)| = \left|f\left(\lim_{k\to\infty} x_{n_k}\right)\right| = \lim_{k\to\infty} |f(x_{n_k})|$$

$$\le \lim_{k\to\infty} \frac{1}{2^{n_k}}|f(x_1)| = 0.$$

So $f(c) = 0$. //

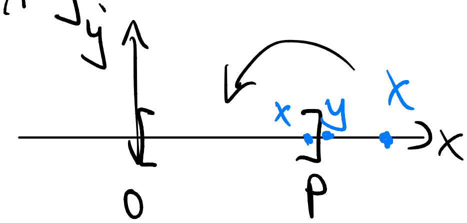
14. Show that a continuous periodic function on $\mathbb{R}$ attains its extreme point and is uniformly continuous on $\mathbb{R}$.

Pf: Since f is periodic, $\exists P \in \mathbb{R}$ s.t. $f(x+P) = f(x)$ for all $x \in \mathbb{R}$.

Extreme values attained: Since f is continuous, $f|_{[0,P]}$ attains max and min in $[0,P]$, say $\exists c_1 \in [0,P]$ s.t. $f(x) \leq f(c_1)$ for all $x \in [0,P]$, $c_2 \in [0,P]$

$f(c_2) \leq f(x)$ for all $x \in [0,P]$.

Let $x \in \mathbb{R}$. Then there is some $n \in \mathbb{N}$ s.t. $x - nP \in [0,P]$



So $f(x) = f(x-nP) \leq f(c_1)$.

$f(x) = f(x-nP) \geq f(c_2)$.

So c_1, c_2 are where max, min of f are attained for all $x \in \mathbb{R}$.

Uniform Cty: f is uniformly continuous on $[-P, 2P]$. So let $\varepsilon > 0$ then $\exists \delta_0 > 0$ s.t. if $x, y \in [-P, 2P]$ with $|x-y| < \delta_0$, then $|f(x) - f(y)| < \varepsilon$.

Now let $x, y \in \mathbb{R}$ with $|x-y| < \delta := \min\{\delta_0, P\}$. Then $\exists n \in \mathbb{N}$ s.t. $x - nP$, $y - nP \in [-P, 2P]$, so

$$|x - nP - (y - nP)| = |x - nP - y + nP| = |x-y| < \delta < \delta_0$$

So $|f(x) - f(y)| = |f(x-nP) - f(y-nP)| < \varepsilon$. //

15. (a) Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are both bounded on A, then their product fg is uniformly continuous on A.

 (b) By considering $f(x) := x$, $g(x) := \sin x$ show that the boundedness assumption is necessary.

Pf: a) Let $\varepsilon > 0$.

$$|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$
$$\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|$$
$$\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \quad (*).$$

Since f, g are bdd, $\exists M > 0$ s.t. $|f|, |g| \leq M$ "take $M = \max\{|f|, |g|\}$."

Then since f, g are unif. cts, $\exists \delta > 0$ s.t. if $|x-y| < \delta$

$$|f(x) - f(y)| < \frac{\varepsilon}{2M}, \quad |g(x) - g(y)| < \frac{\varepsilon}{2M}.$$ "take $\delta = \min\{\delta_1, \delta_2\}$ ..."

Then by $(*)$, we see that

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|$$
$$\leq M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

b) First, show that f, g are uniformly cts.

$f(x) = x$ Trivial.

$g(x) = \sin x$.

sin, cos addition formula.

$$|g(x) - g(y)| = |\sin(x) - \sin(y)| = 2\left|\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right|$$
$$\leq 2\underbrace{\left|\sin\left(\frac{x-y}{2}\right)\right|}_{\leq |x-y|}\underbrace{\left|\cos\left(\frac{x+y}{2}\right)\right|}_{\leq 1}$$

$(|\sin x| \leq |x|)$

$\leq |x-y|.$

So for all $\varepsilon > 0$, taking $\delta = \varepsilon$, we see that g is unif. cts on $\mathbb{R}$.

Now show fg is not unif. continuous. Take $\varepsilon_0 = 2\pi$.

Take $x_n := 2\pi n$, $y_n := 2\pi n + \frac{1}{n}$.

Clearly $\lim_{n\to\infty} (y_n - x_n) = 0$

but

$$|f(x_n)g(x_n) - f(y_n)g(y_n)| = \left|2\pi n \sin 2\pi n - \left(2\pi n + \tfrac{1}{n}\right)\sin\left(2\pi n + \tfrac{1}{n}\right)\right|$$

$$= \left(2\pi n + \tfrac{1}{n}\right)\sin\left(\tfrac{1}{n}\right)$$

$$\geq 2\pi n \sin\left(\tfrac{1}{n}\right) \qquad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1.\right).$$

$$= 2\pi = \varepsilon_0.$$

16. If $f : A \to \mathbb{R}$ is uniformly continuous such that $|f| \geq \sigma > 0$, then $1/f$ is also uniformly continuous.

Pf: Let $\varepsilon > 0$. Let's estimate $\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \left|\frac{f(y)-f(x)}{f(x)f(y)}\right|$

$$\leq |f(y)-f(x)| \frac{1}{|f(x)f(y)|} \quad (*).$$

Since $|f| > \sigma > 0$, $\frac{1}{|f|} < \frac{1}{\sigma}$.

Since f is unif. cts, $\exists \delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon\sigma^2$, so in $(*)$ we see that if $|x-y| < \delta$,

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| \leq |f(x)-f(y)| \frac{1}{|f(x)f(y)|} < \varepsilon\sigma^2 \frac{1}{\sigma^2} = \varepsilon.$$