THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 8

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1. (a) State without proof, the Bolzano-Weierstrass Theorem.

   (b) Show that a sequence (x_n) is convergent if and only if it is Cauchy.

2. Let $a > 0$ and $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. By using monotone convergence theorem, show that (z_n) is convergent and find its limit.

3. Using $\varepsilon - N$, $\varepsilon - \delta$ terminology or the sequential criterion to show that

   (a) $\displaystyle\lim_{n\to+\infty} \frac{n^3+1}{2n^3-3} = \frac{1}{2}$;

   (b) $\displaystyle\lim_{x\to 1} \frac{x^2+2}{x^3-2} = -3$;

   (c) $\displaystyle\lim_{x\to 1} \frac{x}{x^2-1}$ does not exist.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f has a limit L as $x \to 0$. Show that $L = 0$ and f has a limit at every $c \in \mathbb{R}$.

5. Suppose $f : (0, +\infty) \to \mathbb{R}$ is a real valued function such that $\forall \varepsilon > 0$, there exists $\alpha > 0$ such that
$$|f(x) - f(y)| < \varepsilon$$
for all $x, y \in [\alpha, +\infty)$. Show that there is $L \in \mathbb{R}$ such that $\displaystyle\lim_{x\to+\infty} f(x) = L$.

1. (a) State without proof, the Bolzano-Weierstrass Theorem.
   (b) Show that a sequence (x_n) is convergent if and only if it is Cauchy.

Pf: a) Any bounded sequence has a convergent subsequence.

b) Convergence $\Rightarrow$ Cauchy: let $\varepsilon>0$ be given. WTS $\exists N \in \mathbb{N}$ s.t. if $m>n\geq N$ $|x_m - x_n| < \varepsilon$.

Since $x_n \to L$, $\exists N \in \mathbb{N}$ s.t. forall $n \geq N$,

$$|x_n - L| < \frac{\varepsilon}{2}.$$

So for $m>n\geq N$, $|x_m - x_n| = |x_m - L + L - x_n|$

$$\leq |x_m - L| + |L - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Cauchy $\Rightarrow$ Convergence: 1) Cauchy $\Rightarrow$ Bounded: let $\varepsilon = 1$. Then there is an $N \in \mathbb{N}$ s.t. forall $n \geq N$ $|x_n - x_N| < 1$.

So by triangle inequality, we have $|x_n| < 1 + |x_N| < +\infty$

Take $M = \max\{1+|x_N|, |x_1|, \ldots, |x_{N-1}|\}$.

So $|x_n| \leq M$ forall $n \in \mathbb{N}$.

By BW, $\exists (x_{n_k}) \to L$. Let $\varepsilon>0$. Then since x_n is Cauchy, $\exists N_1 \in \mathbb{N}$ s.t. for any $n, m \geq N_1$, $|x_m - x_n| < \frac{\varepsilon}{2}$.

Also, there is a $k_0 \in \mathbb{N}$ s.t. for all $k \geq k_0$

$$|x_{n_k} - L| < \frac{\varepsilon}{2}.$$

For $n \geq \max\{N_1, n_{k_0}\}$, and taking $m = n_{k_0}$, we have

$$|x_n - L| \leq |x_n - x_{n_{k_0}}| + |x_{n_{k_0}} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

2. Let $a > 0$ and $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. By using monotone convergence theorem, show that (z_n) is convergent and find its limit.

Pf: Consider the quadratic $z^2 - z - a = 0$. w/ positive root at
$$z^* = \frac{1+\sqrt{1+4a}}{2}.$$

Case 1: $z_1 < z^*$. Then WTS z_n is increasing

Base Case: $z_2 - z_1 = \sqrt{a+z_1} - z_1 = \dfrac{a+z_1 - z_1^2}{\sqrt{a+z_1}+z_1} > 0.$

Since $z_1 < z^*$, $a + z_1 - z_1^2 > 0$

So $z_2 > z_1$

Sps $z_k > z_{k-1}$. Then $z_{k+1} = \sqrt{a+z_k} > \sqrt{a+z_{k-1}} = z_k$.

So z_n is increasing.

We also show $z_n \le z^*$. By assumption $z_1 < z^*$.

Sps $z_k < z^*$ for some k. Then

$$z_{k+1} = \sqrt{a+z_k} < \sqrt{a + \frac{1+\sqrt{1+4a}}{2}} = \sqrt{\frac{2a+1+\sqrt{1+4a}}{2}} \cdot \sqrt{\frac{2}{2}}$$
$$= \sqrt{\frac{1+2\sqrt{1+4a}+1+4a}{2}} = \frac{\sqrt{(1+\sqrt{1+4a})^2}}{2} = \frac{1+\sqrt{1+4a}}{2} = z^*.$$

Case 2: $z_1 > z^*$. Then show (z_n) is decreasing.

$$z_2 - z_1 = \sqrt{a+z_1} - z_1 = \frac{a+z_1-z_1^2}{\sqrt{a+z_1}+z_1} < 0 \quad \text{b/c. } z_1 > z^*.$$

By induction z_n is decreasing.

Now WTS $z_n \geq \dfrac{1-\sqrt{1+4a}}{2}$. By assumption $z_1 > 0 > \dfrac{1-\sqrt{1+4a}}{2}$.

Similar argument shows $z_n > \dfrac{1-\sqrt{1+4a}}{2}$.

In both cases, we can conclude that (z_n) converges by MCT.

Taking $n \to +\infty$ in the recurrence relation, we get:

$$z = \sqrt{a+z} \iff z^2 - z - a = 0 \Rightarrow z = \frac{1 \pm \sqrt{1+4a}}{2}.$$

So $z = \frac{1+\sqrt{1+4a}}{2}$.

reject negative root.

3. Using $\varepsilon - N$, $\varepsilon - \delta$ terminology or the sequential criterion to show that

   (a) $\lim_{n\to+\infty} \frac{n^3+1}{2n^3-3} = \frac{1}{2}$;

   (b) $\lim_{x\to 1} \frac{x^2+2}{x^3-2} = -3$;

   (c) $\lim_{x\to 1} \frac{x}{x^2-1}$ does not exist.

Pf: a) Let $\varepsilon > 0$ be given.

$$\left| \frac{n^3+1}{2n^3-3} - \frac{1}{2} \right| = \left| \frac{2n^3+2-2n^3+3}{4n^3-6} \right| = \left| \frac{-5}{4n^3-6} \right| \le \left| \frac{5}{4n^3} \right|$$

Then taking $N > \left(\frac{5}{4\varepsilon}\right)^{1/3}$ gives the desired result.

b) Let $\varepsilon > 0$ be given.

$$\left| \frac{x^2+2}{x^3-2} + 3 \right| = \left| \frac{x^2+2+3x^3-6}{x^3-2} \right| = \left| \frac{(x-1)(3x^2+4x+4)}{x^3-2} \right|$$

$$\le |x-1| \left| \frac{3x^2+4x+4}{x^3-2} \right|$$

So for $\frac{1}{2} < x < \frac{3}{2}$. (taking $\delta_1 = 1$)

we have that $\left| \frac{3x^2+4x+4}{x^3-2} \right| \le C$ for some constant C.

does not depend on ε.

So taking $\delta = \min\{\delta_1, \varepsilon\}$, then we have for $0 < |x-1| < \delta$

we have $\left| \frac{x^2+2}{x^3-2} + 3 \right| < \varepsilon$.

c) By subsequential criteria, suffices to choose (x_n) s.t. $x_n \to 1$, $x_n \neq 1$ but $\dfrac{x_n}{x_n^2-1}$ does not converge.

$x_n = 1+\frac{1}{n}$. Clearly $x_n \to 1$, and $x_n \neq 1$ for each n.

But

$$\frac{x_n}{x_n^2-1} = \frac{1+\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2-1} = \frac{1+\frac{1}{n}}{1+\frac{2}{n}+\frac{1}{n^2}-1} = \frac{1+\frac{1}{n}}{\frac{2}{n}+\frac{1}{n^2}} \leq n^2+n$$

which diverges.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f has a limit L as $x \to 0$. Show that $L = 0$ and f has a limit at every $c \in \mathbb{R}$.

Pf: Claim f satisfies:

1) $f(0)=0$: $f(0) = f(0+0) = f(0)+f(0) = 2f(0) \Rightarrow f(0)=0$. ✓
2) $f(-x) = -f(x)$ for $x \in \mathbb{R}$: $0 = f(0) = f(x-x) = f(x)+f(-x)$ ✓
3) for all $n \in \mathbb{N}$, $f(nx) = nf(x)$ shown by induction.
4) for all $n \in \mathbb{N}$. $f(\frac{1}{n}x) = \frac{1}{n}f(x)$:

$$f\left(\frac{x}{n}\right) = 1 \cdot f\left(\frac{x}{n}\right) = \frac{1}{n} \cdot n f\left(\frac{x}{n}\right) = \frac{1}{n} f\left(\cancel{n} \cdot \frac{x}{\cancel{n}}\right) = \frac{1}{n} f(x). ✓$$

Now show $L=0$.

$$f\left(\tfrac{1}{n}\right) = f\left(0 + \tfrac{1}{n}\right) = f\left(\tfrac{1}{n}\right) + f(0)$$
$$= \tfrac{1}{n} f(1) + 0 \to 0 \text{ as } n \to +\infty.$$

Finally, we want to show f has limit at every $c \in \mathbb{R}$. Let $\varepsilon > 0$ be given.
Then since $\lim_{x \to 0} f(x) = 0$, so $\exists\, \delta(\varepsilon) > 0$ s.t. for all x with
$0 < |x| < \delta$, we have $|f(x)| < \varepsilon$.
So for any $0 < |x-c| < \delta$, we have

$$|f(x) - f(c)| = |f(x-c)| < \varepsilon.$$

So $\lim_{x \to c} f(x) = f(c)$. //

5. Suppose $f : (0, +\infty) \to \mathbb{R}$ is a real valued function such that $\forall \varepsilon > 0$, there exists $\alpha > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

for all $x, y \in [\alpha, +\infty)$. Show that there is $L \in \mathbb{R}$ such that $\lim_{x \to +\infty} f(x) = L$.

Pf: WTS $\exists (x_n)$ s.t. $f(x_n)$ is Cauchy,

For each n, $\exists \alpha_n \in \mathbb{R}$ s.t. for all $x, y \geq \alpha_n$

$|f(x) - f(y)| < \frac{1}{n}$.

We let $x_n = \alpha_n$.

We know that any sequence admits a monotonic subsequence.

So WLOG, we can take x_n to be increasing.

So let $\varepsilon > 0$ be given. Then by A.P. $\exists N \in \mathbb{N}$.

s.t. $\frac{1}{N} < \varepsilon$, and for this N, there is α_N s.t. for all $x, y \geq \alpha_N$, $|f(x) - f(y)| < \varepsilon$

So in particular, for $n, m \geq N$, $\alpha_n, \alpha_m \geq \alpha_N$.

so that $|f(\alpha_n) - f(\alpha_m)| < \varepsilon$.

So $(f(\alpha_n))$ is Cauchy and hence converges to some limit, say L. Let $\varepsilon > 0$ be given. Then there is an $\alpha \in \mathbb{R}$ s.t. for all $x, y \geq \alpha$, $|f(x) - f(y)| < \frac{\varepsilon}{2}$.

By convergence of $(f(x_n))$, $\exists N \in \mathbb{N}$, s.t. $|f(x_n) - L| < \frac{\varepsilon}{2}$.

So for $x \geq \max\{N, \alpha\}$, we have

$$|f(x) - L| \leq |f(x) - f(x_n)| + |f(x_n) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$