MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 10

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Key Concepts

Let $T : V \to V$ be a linear operator on V over $\mathbb{C}$, where $\dim(V) < \infty$.

1. **Jordan Canonical Form**
   - T is diagonalizable if f_T splits and $\gamma_T(\lambda_i) = \mu_T(\lambda_i)$ for all i.
   - There exists a basis β such that

$$[T]_\beta = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \text{ where } A_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

     - λ_i are the eigenvalues of T (not necessarily distinct).
     - $J = [T]_\beta$ is called the *Jordan canonical form* of T.
     - A_i is called a *Jordan block* corresponding to λ_i.
     - β is called the *Jordan canonical basis*.
   - For a Jordan block A of size k with eigenvalue λ:
     - A has only one eigenvalue λ.
     - $\dim(E_\lambda) = 1$.
     - The smallest $p \geq 1$ such that $(A - \lambda I)^p$ is k, so $N((A - \lambda I)^p) = \mathbb{C}^k$.
     - If $\{e_1, ..., e_k\}$ is the standard basis for $\mathbb{C}^k$, then $(A - \lambda I)^i e_i = 0$ for $1 \leq i \leq k$.
   - *Generalized eigenvector*: $x \in V \backslash \{0\}$ such that $(A - \lambda I)^p x = 0$ for some $p \geq 1$.
   - *Generalized eigenspace*: $K_\lambda := \{x \in \mathbb{C}^n : (A - \lambda I)^p x = 0 \text{ for some } p \geq 1\}$.
   - *Jordan Decomposition Theorem*: Let $A \in M_{n\times n}(\mathbb{C})$ with distinct eigenvalues $\lambda_1, ..., \lambda_k$ with corresponding multiplicities $m_1, ..., m_k$. Then
     - $\dim(K_{\lambda_i}) = m_i$.
     - $\mathbb{C}^n = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$.
     - Each K_{λ_i} has a basis $\beta_i = \gamma_{1,i} \cup \cdots \cup \gamma_{l,i}$, where every $\gamma_{m,i}$ is a *cycle*:

$$\gamma_{m,i} = \{(A - \lambda_i I)^{p-1} x, (A - \lambda_i I)^{p-2} x, ..., (A - \lambda_i I)x, x\}.$$

     - Please refer to the lecture notes for a detailed proof.

Exercises

1. Let U be a linear operator on a finite-dimensional vector space V. Prove the following results.

   (a) $N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$.

   (b) If $\operatorname{rank}(U^m) = \operatorname{rank}(U^{m+1})$ for some positive integer m, then $\operatorname{rank}(U^m) = \operatorname{rank}(U^k)$ and $N(U^m) = N(U^k)$ for any positive integer $k \geq m$.

   (c) Let T be a linear operator on V, and let λ be an eigenvalue of T. Prove that if $\operatorname{rank}((T-\lambda I)^m) = \operatorname{rank}((T-\lambda I)^{m+1})$ for some integer m, then $K_\lambda = N((T-\lambda I)^m)$.

   (d) *Second Test for Diagonalizability.* Let T be a linear operator on V whose characteristic polynomial splits, and let $\lambda_1, \lambda_2, ..., \lambda_k$ be the distinct eigenvalues of T. Then T is diagonalizable if and only if $\operatorname{rank}(T - \lambda_i I) = \operatorname{rank}((T - \lambda_i I)^2)$ for $1 \leq i \leq k$.

   (e) Use (d) to prove that if T is a diagonalizable linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then T_W is diagonalizable.

2. Let V be the real vector space of functions spanned by the set of real-valued functions $\{1, t, t^2, e^t, te^t\}$, and T the linear operator on V defined by $T(f) = f'$. Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.

3. Let $\gamma_1, \gamma_2, ..., \gamma_p$ be cycles of generalized eigenvectors of a linear operator T corresponding to an eigenvalue λ. Prove that if the initial eigenvectors are distinct, then the cycles are disjoint.