MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 9

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Key Concepts

Let $T : V \to V$ be a linear operator on V over F, where $F = \mathbb{R}$ or $\mathbb{C}$ and $\dim(V) < \infty$.

1. **Unitary and orthogonal operators**
   - T is *unitary* (if $F = \mathbb{C}$) or *orthogonal* (if $F = \mathbb{R}$) if $\|T(x)\| = \|x\|$ for all $x \in V$.
   - The condition ($\|T(x)\| = \|x\|$ for all $x \in V$) is equivalent to the following:
     (a) $TT^* = T^*T = I$. *(Used to define unitary and orthogonal matrices.)*
     (b) T preserves the inner product on V: $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.
     (c) $T(\beta) := \{T(v_1), ..., T(v_n)\}$ is an o.n. basis for V for any o.n. basis β for V.
     (d) $\exists$ an o.n. basis β for V such that $T(\beta)$ is an o.n. basis for V.
     (e) $\exists$ an o.n. basis β such that $[T]_\beta$ is unitary (resp. orthogonal).
   - Let $v_1, ..., v_n \in F^n$. Then $A = [v_1, ..., v_n] \in M_{n\times n}(F)$ is unitary (or orthogonal) iff $\{v_1, ..., v_n\}$ is an o.n. basis for $\mathbb{C}^n$ (resp. $\mathbb{R}^n$).
   - $A \in M_{n\times n}(\mathbb{C})$ is *unitarily equivalent* if $\exists\ P \in U(n)$ s.t. P^*AP is diagonal.
   - $A \in M_{n\times n}(\mathbb{R})$ is *orthogonally equivalent* if $\exists\ P \in O(n)$ s.t. P^TAP is diagonal.
   - When $F = \mathbb{C}$, A is normal iff A is unitarily equivalent to a diagonal matrix.
   - When $F = \mathbb{R}$, A is symmetric iff A is orthogonally equivalent to a diagonal matrix.

2. **Spectral decomposition**
   - *Orthogonal projection on W*: $T(y) = \sum_{i=1}^k \langle y, v_i\rangle\, v_i$, which is linear and satisfies:
     - $N(T) = W^\perp$ and $R(T) = W$.
     - $T^2 = T$.
     - T is self-adjoint.
   - *Spectral decomposition*: Let T is normal (resp. self-adjoint) and has distinct eigenvalues $\lambda_1, ..., \lambda_k$ *(Spectrum of T)*. Let T_i be the orthogonal projection on $E_i := E_{\lambda_i}$.
     (a) $V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$
     (b) $E_i^\perp = \bigoplus_{j\neq i} E_j$ for all $1 \le i \le k$
     (c) $T_iT_j = \delta_{ij}T_i$ for all $1 \le i, j \le k$
     (d) $I = T_1 + T_2 + \cdots + T_k$
     (e) $T = \lambda_1T_1 + \lambda_2T_2 + \cdots + \lambda_kT_k$
   - If $F = \mathbb{C}$, then T is normal iff $T^* = g(T)$ for some polynomial g.

Exercises

1. Let A and B be $n \times n$ matrices that are unitarily equivalent.

   (a) Prove that $\operatorname{tr}(A^*A) = \operatorname{tr}(B^*B)$.

   (b) Prove that
   $$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2.$$

   (c) Show that the matrices
   $$\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$$
   are not unitarily equivalent.

   (d) Prove that A is positive definite (or semidefinite) if and only if B is positive definite (resp. semidefinite).

2. Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

3. Prove that if T is a normal operator on a complex finite-dimensional inner product space and U is a linear operator that commutes with T, then U commutes with T^*.