MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 8

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Key Concepts

Let $T : V \to V$ be a linear operator on V over F, where $F = \mathbb{R}$ or $\mathbb{C}$ and $\dim(V) < \infty$.

1. **Adjoint of a linear operator**
   - For any $g \in \mathcal{L}(V, F)$, $\exists!\ y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$.
   - *Adjoint of* T: $\exists!\ T^* \in \mathcal{L}(V)$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.
   - Suppose β is an orthonormal basis for V, then $[T^*]_\beta = ([T]_\beta)^*$.
   - If T has an eigenvector (corr. to λ), then T^* has an eigenvector (corr. to $\overline{\lambda}$).
   - Please refer to lecture notes for the basic properties of adjoint.

2. **Existence of orthonormal eigenbasis**
   - *Schur decomposition*: If f_T splits, then $\exists$ o.n. basis β s.t. $[T]_\beta$ is upper triangular.
   - T is *normal* if $TT^* = T^*T$. Examples:
     - T is *unitary* ($F = \mathbb{C}$) or *orthogonal* ($F = \mathbb{R}$) if $T^*T = TT^* = I$.
     - T is *Hermitian* (or *self-adjoint*) if $T^* = T$.
     - T is *skew-Hermitian* (or *anti-self-adjoint*) if $T^* = -T$.
   - Some properties when T is normal:
     - $\|T(x)\| = \|T^*(x)\|$
     - $T - cI$ is normal for all $c \in F$
     - If $T(x) = \lambda x$, then $T^*(x) = \overline{\lambda} x$
     - If λ_1, λ_2 are distinct eigenvectors of T (corr. $x_1, x_2 \in V$), then x_1, x_2 orthogonal.
   - Some properties when T is self-adjoint:
     - Every eigenvalue of T is real.
     - If $F = \mathbb{R}$, then f_T splits.
   - When $F = \mathbb{C}$, T is normal iff there exists orthonormal eigenbasis of T for V.
   - When $F = \mathbb{R}$, T is self-adjoint iff there exists orthonormal eigenbasis of T for V.

Exercises

1. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

   (a) $R(T^*)^\perp = N(T)$.

   (b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$.

2. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following results.

   (a) $N(T^*T) = N(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.

   (b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce that $\text{rank}(TT^*) = \text{rank}(T)$.

3. Let T be a normal operator on a finite-dimensional real inner product space V. If f_T splits, prove that T is self-adjoint.