MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 7

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Key Concepts

Let $T : V \to V$ be a linear operator on V over F, where $F = \mathbb{R}$ or $\mathbb{C}$.

1. **Inner product & norm**
   - *Inner product*: A map $\langle \cdot, \cdot \rangle : V \times V \to F$ such that for all $x, y, z \in V$ and $c \in F$:
     (a) $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$
     (b) $c\langle x, y\rangle = \langle cx, y\rangle$
     (c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$
     (d) $\langle x, x\rangle > 0$ if $x \neq 0$ (i.e.: $\langle x, x\rangle \in \mathbb{R}_{\geq 0}$)
   - *Norm*: $\|x\| := \sqrt{\langle x, x\rangle}$ (see Exercise Q1 for the properties of norm)
2. **Orthogonality**
   - $S \subset V$ is *orthogonal* if $\langle x, y\rangle = 0$ for all $x, y \in S$; *orthonormal* if $\|x\| = 1$ in addition.
   - Orthogonal sets containing non-zero vectors are linearly independent.
   - If $S = \{v_1, ..., v_k\}$ is an orthogonal set in V, then for all $y \in \text{span}(S)$,
   $$y = \sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{\|v_i\|^2}\, v_i \quad \left(= \sum_{i=1}^{k} \langle y, v_i\rangle\, v_i \;\text{ if } S \text{ is orthonormal}\right).$$
   - If V has an o.n. basis $\beta = \{v_1, ..., v_n\}$, then $([T]_\beta)_{ij} = \langle T(v_j), v_i\rangle v_i$.
   - *Gram-Schmidt process*: Generate an orthogonal set $S'_n = \{v_1, ..., v_n\}$ from a linearly independent set $S_n = \{w_1, ..., w_n\}$, which satisfies $\text{span}(S'_n) = \text{span}(S_n)$:
     - Set $v_1 = w_1$, then for $k > 1$, let $v_k = w_k - \displaystyle\sum_{i=1}^{k-1} \frac{\langle y, v_i\rangle}{\|v_i\|^2}\, v_i$.
     - Normalizing in each step allows simpler calculation and yields an o.n. basis.
   - *Orthogonal complement*: $S^\perp := \{x \in V : \langle x, y\rangle = 0 \ \ \forall\, y \in S\}$, $S \subset V$ nonempty. Let W be a finite-dim subspace of V, $\beta = \{v_1, ..., v_k\}$ be an o.n. basis for W:
     - For all $y \in V$, there exists unique $u \in W$ and $z \in W^\perp$ such that $y = u + z$.
     - Furthermore, $u = \displaystyle\sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{\|v_i\|^2}\, v_i$, so $\|y - x\| \geq \|y - u\|$ for all $x \in W$.
     - If $\dim(V) < \infty$, β can be extended to o.n. basis β' for V, then $\beta' \backslash \beta$ is an o.n. basis for $W^\perp$. So $\dim(V) = \dim(W) + \dim(W^\perp)$ and $W \cap W^\perp = \{0\}$.

Exercises

1. Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, we have

   (a) $\|cx\| = |c| \cdot \|x\|$

   (b) $\|x\| \geq 0$, and $\|x\| = 0$ iff $x = 0$

   (c) *Cauchy-Schwarz Inequality*: $|\langle x, y\rangle| \leq \|x\|\|y\|$

   (d) *Triangle inequality*: $\|x + y\| \leq \|x\| + \|y\|$

   We proved (a) and (b) in the lecture. Prove (c) and (d) now.
   Furthermore, when will the equalities hold?

2. Let $V = C([0, 1])$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

   Let W be the subspace spanned by the linearly independent subset $\{t, \sqrt{t}\}$.

   (a) Find an orthonormal basis for W.

   (b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the best (closest) approximation of h in W.

3. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in W^{\perp}$ such that $\langle x, y \rangle \neq 0$.

4. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.

   (a) $S_0 \subseteq S$ implies that $S^{\perp} \subseteq S_0^{\perp}$.

   (b) $S \subseteq (S^{\perp})^{\perp}$; so $\mathrm{span}(S) \subseteq (S^{\perp})^{\perp}$.

   (c) $W = (W^{\perp})^{\perp}$.

   (d) $V = W \oplus W^{\perp}$.