MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 6

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Key Concepts

Let $T : V \to V$ be a linear operator on V for the following section.

1. **Invariant subspaces**

   - *T-invariant subspace*: A subspace $W \in V$ such that $T(W) \subseteq W$.
     - E.g.: $\{0\}$, V, $R(T)$, $N(T)$, E_λ
     - Let $\lambda_1, ..., \lambda_k$ be distinct eigenvalues of T. Is $\bigoplus E_{\lambda_i}$ T-invariant?
     - Define $T_W : W \to W$ by $T_W(w) = T(w)$. Then f_{T_W} divides f_T.
   - *T-cyclic subspace generated by x*: $W = \text{span}(\{x, T(x), T^2(x), ...\} \subseteq V$.
     - W is the smallest T-invariant subspace containing x.
     - Let $k = \dim(W) < \infty$, then $\{x, T(x), T^2(x), ..., T^k(x)\}$ is a basis for W.
     - If $a_0x + a_1T(x) + a_2T^2(x) + ... + a_{k-1}T^{k-1}(x) + T^k(x) = 0$, then $f_{T_W}(t) = (-1)^k(a_0 + a_1t + a_2t^2 + ... + a_{k-1}t^{k-1} + t^k)$.
   - Cayley-Hamilton Theorem: Suppose $\dim(V) < \infty$, then $f_T(T) = T_0$.

Exercises

1. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, ..., v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in W$ for all i.

2. Let T be a linear operator on an n-dimensional vector space V such that T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself (i.e. there exists $x \in V$ such that V is the T-cyclic subspace generated by x).

3. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.