MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 5

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Key Concepts

Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional VS over F. In addition, let $\beta = \{v_i\}_{i=1}^n$ and $\gamma = \{v_i\}_{i=1}^m$ be ordered bases of V and W respectively.

1. **Dual space (finite-dimensional)**
   - *Dual space*: $V^* \in \mathcal{L}(V, F)$ is the space of all linear functional on V.
   - *Dual basis*: $\beta^* = \{f_1, ..., f_n\} \subset V^*$ such that $f_j(v_i) = \delta_{ij}$.
   - *Dual map*: $T^* : W^* \to V^*$ defined by $T^*(g) = g(T)$, then $[T^*]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^T$.
   - *Double dual*: $V^{**} = (V^*)^*$ with basis $\beta^{**} = (\beta^*)^*$.
   - V and V^{**} isomorphic: $l : V \to V^{**}$ defined by $l(v)(f) = f(v)$.

Let $T : V \to V$ be a linear operator on V for the following sections.

2. **Eigenvalues and eigenvectors**
   - T is *diagonalizable* if $\dim(V) < \infty$ and $[T]_\beta$ is diagonal for some ordered basis $\beta \subset V$.
   - $v \in V \backslash \{0\}$ is an *eigenvector associated with eigenvalue* $\lambda \in F$ if $T(v) = \lambda v$.
   - T is diagonalizable iff there exists an ordered basis consisting of eigenvectors of T.
   - *Characteristic polynomial*: $f_T(t) = \det([T]_\beta - tI_n) \in P_n(F)$, which is well-defined, i.e. independent of the choice of β as $[T]_\beta = Q^{-1}[T]_{\beta'}Q$, where $Q = [I_V]_{\beta'}^\beta$.
   - λ is an eigenvalue of T iff $f_T(t) = 0$ as $T(v) = \lambda v$ iff $v \in N(T - \lambda I_V)$.

3. **Eigenspaces and multiplicities**
   - *Eigenspace*: $E_\lambda := N(T - \lambda I_V) = \{v \in V : T(v) = \lambda(v)\} \subset V$.
   - *Algebraic multiplicity*: $\mu_T(\lambda) := \max\{k \in \mathbb{N} : (t - \lambda)^k \mid f_T(t)\}$.
   - *Geometric multiplicity*: $\gamma_T(\lambda) := \dim(E_\lambda)$, then $1 \le \gamma_T(\lambda) \le \mu_T(\lambda)$.
   - If $\lambda_1, ..., \lambda_k$ are eigenvalues of T and S_i are linearly independent subsets of E_{λ_i}, then $\bigcup_i S_i$ is linearly independent.
   - Suppose T has distinct eigenvalues $\lambda_1, ..., \lambda_k$, $\dim(V) < \infty$, and $f_T(t)$ splits, then T is diagonalizable iff $\gamma_T(\lambda) = \mu_T(\lambda)$ for all i.
     In such case, $\beta = \bigcup_i \beta_i$ is an eigenbasis of T, where β_i are ordered bases of E_{λ_i}.

Exercises

1. Let V be a finite-dimensional vector space. Recall that the map $l : V \to V^{**}$ defined by $l(v)(f) = f(v)$ is an isomorphism. Prove that $l(\beta) = \beta^{**}$.

2. Let V and W be nonzero vector spaces over the same field, and let $T : V \to W$ be a linear transformation.

   (a) Prove that T is onto if and only if T^* is one-to-one.

   (b) Prove that T^* is onto if and only if T is one-to-one.

3. Let $T : V \to V$ be a diagonalizable linear operator, and let $f, g \in P(F)$, prove the following statements.

    (a) $f(T)$ and $g(T)$ is simultaneously diagonalizable, i.e. there exists an ordered basis β for V such that both $[f(T)]_\beta$ and $[g(T)]_\beta$ are diagonal.

    (b) $f(T)$ and $g(T)$ commute, i.e. $f(T) \circ g(T) = g(T) \circ f(T)$.

4. Let T be a linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^T$.

    (a) Show that ± 1 are the only eigenvalues of T.

    (b) Describe the eigenvectors corresponding to each eigenvalue of T.

    (c) Find an ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

    (d) Find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix for $n > 2$.