MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 4

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Key Concepts

We assume $T : V \to W$ is a linear transformation unless otherwise stated.
In addition, let β and γ be ordered bases of V and W respectively.

1. **Invertibility and isomorphism**

   - If T invertible (i.e. T^{-1} exists), then $\dim(V) < \infty$ iff $\dim(W) < \infty$.
     In this case, $\dim(V) = \dim(W)$.
   - Suppose V and W are finite-dimensional (so that $[T]_\beta^\gamma$ is a well-defined matrix).
     Then T is invertible iff $[T]_\beta^\gamma$ is invertible, and $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.
   - V is *isomorphic* to W if there exists an invertible linear transformation $T : V \to W$.
     Then T is an *isomorphism* from V onto W.
   - Suppose $\dim(V), \dim(W) < \infty$. Then V is isomorphic to W iff $\dim(V) = \dim(W)$.

2. **Space of linear transformations**

   - Denote $\mathcal{L}(V, W)$ as the space of all linear transformations from V to W.
   - Suppose V and W are finite-dimensional (so that $[T]_\beta^\gamma$ is a well-defined matrix).
     Then $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.
     So $\dim(\mathcal{L}(V, W)) = \dim(V) \cdot \dim(W)$.

3. **Change of coordinates**

   - Let $I_V : V \to V$ be the identity map on V, and β, β' be ordered bases of V.
     Then $Q = [I_V]_{\beta'}^\beta$ is the *change of coordinate matrix* from β' to β as

     $$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^\beta [v]_{\beta'} = Q[v]_{\beta'}.$$

   - Note that I_V is invertible $\Rightarrow$ Q is invertible, and $Q^{-1} = [I_V]_\beta^{\beta'}$. (Why?)
     Then
     $$[T]_\beta = [I_V \circ T \circ I_V]_\beta = [I_V]_{\beta'}^\beta [T]_{\beta'} [I_V]_\beta^{\beta'} = Q^{-1}[T]_{\beta'}Q.$$
   - $[T]_\beta$ and $[T]_{\beta'}$ are similar.

Exercises

1. Prove the following linear transformations are isomorphisms.

    (a) $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ defined by $\Phi(A) = B^{-1}AB$.

    (b) $T : P_n(F) \to F^{n+1}$ defined by $T(f) = (f(c_0), f(c_1), ..., f(c_n))$, where $c_0, c_1, ..., c_n \in F$ are distinct scalars, and F is an infinite field.

2. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V, W) : T(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

    (a) S^0 is a subspace of $\mathcal{L}(V, W)$.

    (b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

    (c) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

3. Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, ..., x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, x'_2, ..., x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix from β' to β.

4. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a, b) = (2a + b, a - b)$. Let $\beta_1 = \{(-1, 3), (2, -1)\}$ and $\beta_2 = \{(0, 2), (1, 0)\}$ be ordered bases of $\mathbb{R}^2$, and β be the standard ordered basis of $\mathbb{R}^2$.

   (a) Find $[T]_\beta$, $[I]^\beta_{\beta_1}$, $[I]^\beta_{\beta_2}$, and $[I]^{\beta_2}_{\beta_1}$.

   (b) Express $[T]_{\beta_1}$, $[T]_{\beta_2}$, and $[T]^{\beta_2}_{\beta_1}$ as the matrices above. Compute them for practice.