MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 3

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Key Concepts

We assume $T : V \to W$ is a linear transformation unless otherwise stated.

1. **Linear transformation: Null space and range**

   - Null space $N(T) := \{v \in V : T(v) = 0\}$ is a subspace of V, $\text{Nullity}(T) = \dim(N(T))$.
   - Range $R(T) := \{T(v) : v \in V\}$ is a subspace of W, $\text{Rank}(T) = \dim(R(T))$
   - (Rank-Nullity Theorem) $\dim(V) = \text{nullity}(T) + \text{rank}(T)$ if $\dim(V) < \infty$.
     Remarks: W could be infinite-dimensional!

2. **Matrix representation of linear transformation**

   - Let $\beta = \{v_j\}_{1 \le j \le n}$ and $\gamma = \{w_i\}_{1 \le i \le m}$ be ordered bases of V and W respectively.
   - If $v = \sum_{j=1}^{n} \alpha_j v_j$, then $[v]_\beta = (\alpha_1, ..., \alpha_n)^T \in F^n$.
   - If $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$, then

$$[T]_\beta^\gamma = \big(a_{ij}\big)_{\substack{1 \le i \le m \\ 1 \le j \le n}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix} \in F^{m \times n}.$$

   - $[T(v)]_\gamma$ is a linear combination of column vectors of $[T]_\beta^\gamma$ with coefficients $[v]_\beta$:

$$[T(v)]_\gamma = \Big[\sum_{j=1}^{n} \alpha_j T(v_j)\Big]_\gamma = \sum_{j=1}^{n} [T(v_j)]_\gamma \cdot \alpha_j = [T]_\beta^\gamma \ [v]_\beta$$

3. **Composition of linear transformation**

   - Let $T : V \to W$ and $U : W \to Z$ be linear transformations, α, β, γ be the ordered bases of V, W, Z respectively, then UT is linear, and

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma \ [T]_\alpha^\beta \in F^{|\gamma| \times |\alpha|}$$

Exercises

1. Let $\beta = \{1, 1+x, 1+x^2, 1+x^3\}$ be an ordered basis of $P_3(\mathbb{R})$, and $\gamma = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ be the standard ordered basis of $\mathbb{R}^{2\times 2}$. Define $T : P_3(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ by

$$T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(1) \end{pmatrix}.$$

   (a) Find $[T]_\beta^\gamma$ and $N(T)$.

   (b) Let $f(x) = 1 - 3x + 2x^3$, find $[T(f)]_\gamma$.

   (c) Let $U : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be a linear transformation defined by $U(B) = AB$, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

   Find $[U]_\gamma$ and $[UT]_\beta^\gamma$. What is $N(UT)$?

2. Let V and W be vector spaces such that $\dim(V) = \dim(W) < \infty$, and let $T : V \to W$ be linear. Show that there exist bases β and γ for V and W respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

3. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

   (a) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$.

   (b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.