MATH2048: Honours Linear Algebra II
2024/25 Term 1

Tutorial 2

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Key Concepts

1. **Quotient spaces**
   - $V/W = \{v + W : v \in V\}$, where $v + W = \{v + w : w \in W\}$ is called a coset of W in V.
   - $v + W$ is a subset of V. Moreover, it is a subspace of V if and only if $v \in W$.
     (Prove it in HW 2 Q2(a))

2. **Existence of basis using Zorn's Lemma**
   - Zorn's Lemma: Let S be a partially ordered set. If every chain $\mathcal{C}$ of S has an upper bound in S, then S contains a maximal element.
     (Why only require the upper bound to be in S but not $\mathcal{C}$?)
   - Let S be the collection of linearly independent subsets of V. Then S is partially ordered under $\subseteq$, and for all chain $\mathcal{C} = \{L_\alpha\}_{i \in I}$ in S, we know that $\bigcup_\alpha L_\alpha$ is an upper bound of $\mathcal{C}$ and is linearly independent (i.e. in S) (Why?).
   - Then, there exists a maximal element M which is linearly independent and spans V. So, M is a basis of V.

3. **Linear transformation**
   - A map $T : V \to W$ is linear if it satisfies the following:
     - Addition:
     - Multiplication:
   - Some properties:
     - $T(0_V) = 0_W$
     - $T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i)$

Exercises

1. Prove the following generalization of the replacement theorem: Let β be a basis for a vector space V, and let S be a linearly independent subset of V. Then there exists a subset S_1 of β such that $S \cup S_1$ is a basis for V.

2. Recall that in Lecture Notes 5 P. 7, we try to prove there exists a minimal spanning set of V inside a collection $\mathcal{C}$ of all spanning sets via Zorn's Lemma. The proof is invalid as the infinite intersection of decreasing spanning sets in a chain may not be a spanning set. Construct an example for that.

3. Let $C(\mathbb{R})$ be the space of real-valued continuous functions. Let a, b be real numbers such that $a < b$. Define $T : C(\mathbb{R}) \to \mathbb{R}$ by

$$T(f) = \int_a^b f(x)g(x)dx$$

where $g \in C(\mathbb{R})$. Prove that T is a linear transformation.

4. Let V be a vector space and U, W be subspaces of V such that $U \oplus W = V$. Define $T : U \to V/W$ by $T(u) = u + W$. Prove that T is a well-defined linear transformation. Also, prove that T is bijective.