# MATH2048: Honours Linear Algebra II
## 2024/25 Term 1

# Tutorial 1

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## Key Concepts

1. **Vector spaces and subspaces (over a field $\boldsymbol{F}$)**
   - Closed under addition and scalar multiplication (What are VS1 - VS8? Why?)
   - How to prove a subspace? How to disprove a subspace?

   - Are they vector spaces?
     (a) $\mathbb{R}^n$ over $\mathbb{R}$, $\mathbb{Q}^n$ over $\mathbb{Q}$, $\mathbb{Z}^n$ over $\mathbb{Z}$
     (b) $\mathbb{C}^n$ over $\mathbb{R}$, $\mathbb{R}^n$ over $\mathbb{Q}$, $\mathbb{Z}^n$ over $\mathbb{F}_p$

2. **Linear combinations, span, linear independence, bases and dimensions**
   - Linear combination: $a_1v_1 + ... + a_nv_n$
   - Linear independence: $a_1v_1 + ... + a_nv_n = 0$ implies $a_i = 0$ for all $i$
     If a nontrivial solution exists, then $v_1, ..., v_n$ are linearly dependent.
   - $\text{span}(\{v_1, ..., v_n\}) = \{a_1v_1 + ... + a_nv_n : a_i \in F\}$
   - $\beta$ is a basis of $V$ if $\text{span}(\beta) = V$ and $\beta$ is linearly independent.
     Then $\dim(V) = |\beta|$.
   - Prove that $\{1, 1 + x, 1 + x + x^2\}$ is a basis of $P_2(x)$.

   - Prove that $\{10 + x, 27 + 2x^2, -6 + 5x - 1.5x^2, x\log 3 - 7x^2\}$ is not a basis of $P_2(x)$.

3. **Sums, direct sums and products**
   - $U + V =$
   - $U \times V =$
   - What are the differences between $U \cup V$, $U + V$, $U \oplus V$ and $U \times V$?
   - Is $\{e_k\}_{k=1}^{\infty}$ a basis for $\bigoplus_{k=1}^{\infty} F$ and $\prod_{k=1}^{\infty} F$?

# Exercises

1. Given that $V$, the set of sequences $\{a_n\}$ of real numbers, is a vector space. Show that the set of convergent sequences $\{a_n\}$ is a subspace of $V$.

2. Let $V$ be a vector space over a field of characteristic not equal to 2. If $u, v, w \in V$, show that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, v + w, w + u\}$ is linearly independent.

3. Suppose $n$ is a positive integer. For $0 \leq k \leq n$, let

$$p_k(x) = x^k(1-x)^{n-k}.$$

Show that $p_0, ..., p_n$ is a basis of $\mathcal{P}_n(F)$.

4. Let $V$ be a vector space with dimension $n$. Suppose $d_1, ..., d_k \geq 1$ be integers such that $\sum_{i=1}^k d_i = n$. Prove that there exist subspaces $U_1, ..., U_k$ such that

$$\bigoplus_{i=1}^k U_i = V$$
$$\dim(U_i) = d_i.$$