MATH2048: Honours Linear Algebra II
2024/25 Term 1

Midterm Examination 2

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let T_n be a linear operator on $\mathbb{R}^n$ for all positive integers $n \geq 2$. Define T_n by

$$T_n(e_i) = e_{i+1} \quad \text{for all } i = 1, 2, ..., n,$$

where $\{e_i\}$ is the standard ordered basis of $\mathbb{R}^n$ and we write $e_{n+1} = e_1$.

   (a) Find all the eigenvalues and corresponding eigenspaces of T_2, then determine whether T_2 is diagonalizable.

   (b) Prove that T_n is not diagonalizable for all $n \geq 3$.

2. Let T be a linear operator on $P_2(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = a(-1 + 3x + 3x^2) + b(3x + x^2) + c(-x + x^2).$$

   (a) Find a polynomial $f \in P_2(\mathbb{R})$ such that $T^3 = f(T)$.

   (b) Let $g(x) = 3x + 2x^2$ and W is the T-cyclic subspace generated by g. Find $\dim(W)$ and the characteristic polynomial of T_W.

3. Let V and W be nonzero vector spaces over the same field, and let $T : V \to W$ be a linear transformation. Prove that the dual map $T^* : W^* \to V^*$ is onto if and only if T is one-to-one.

4. Let T and U be diagonalizable linear operators on a finite-dimensional vector space V. Prove that if $UT = TU$, then UT is diagonalizable.
Hint: Prove that U and T are simultaneously diagonalizable.

5. Suppose T is a linear operator on a finite-dimensional vector space V.

   (a) Prove that if λ is an eigenvalue of T, then λ^m is an eigenvalue of T^m for all positive integers $m \geq 2$.

   (b) Prove the converse of (a): If λ^m is an eigenvalue of T^m for all positive integers $m \geq 2$, then λ is an eigenvalue of T.
   Hint: Suppose the statement is not true, prove that for all coprime positive integers x, y (i.e. $\gcd(x, y) = 1$), if $T^x(v_1) = \lambda^x v_1$ and $T^y(v_2) = \lambda^y v_2$ for some non-zero vectors $v_1, v_2 \in V$, then v_1, v_2 are linearly independent.
   You may also use this fact without proof: For all coprime positive integers x, y, there exist positive integers a, b such that $ax - by = 1$.

   (c) Prove or disprove the converse of (a) if V is infinite-dimensional.

END OF PAPER