MATH2048: Honours Linear Algebra II
2024/25 Term 1

Midterm Examination 1

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be defined by $T(f(x)) = f(x) - f'(x)$.

   (a) Prove that T is linear, and it is an isomorphism.

   (b) Consider $n = 2$, and let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, 1+x, 1+x+x^2\}$ be ordered bases of $P_2(\mathbb{R})$. Let $f(x) = x^2 - 2x + 4$. Find $[f]_\beta$, then compute $[f]_\gamma$ using the change of coordinate matrix.

   (c) Is it true that $[T]_\beta[f]_\beta = [T]_\gamma[f]_\gamma$? Please explain your answer with all the details.

2. Given the matrix
$$A = \begin{pmatrix} 3 & 3 & 9 \\ -6 & 3 & 0 \\ -1 & \frac{-5}{2} & -6 \end{pmatrix}.$$
   Define the linear transformation $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ by $L_A(x) = Ax$.

   (a) Find $R(L_A)$ and $N(L_A)$, then determine whether or not $\mathbb{R}^3 = R(L_A) \oplus N(L_A)$.

   (b) Define $\widetilde{L_A} : M_{3\times 3}(\mathbb{R}) \to M_{3\times 3}(\mathbb{R})$ by $\widetilde{L_A}(B) = AB$. Find $\operatorname{rank}(\widetilde{L_A})$ and $\operatorname{nullity}(\widetilde{L_A})$.

3. Let T_1 and T_2 be linear maps from V to W, where V and W are vector spaces over $\mathbb{F}$. Suppose both V and W are finite-dimensional. Prove that $\operatorname{nullity}(T_1) = \operatorname{nullity}(T_2)$ if and only if there exists invertible linear maps $R : V \to V$ and $S : W \to W$ such that $T_1 = ST_2R$.

4. Let V be a vector space, and W be a subspace of V. Using Zorn's Lemma, prove that there is a linearly independent subset $\beta \subset V$, such that $V = \operatorname{span}(\beta) \oplus W$. Please explain your answer with all the details.

5. Let V be the subspace of $\mathbb{R}^\infty$ defined by
$$V = \{(x_1, x_2, ...) \in \mathbb{R}^\infty : x_k \neq 0 \text{ for finitely many } k\},$$
   and for all positive integers n, let W_n be subsets of V defined by
$$W_n = \{(x_1, x_2, ...) \in V : \sum_{k=1}^{n} k^2 x_k = 0 \text{ and } x_i = 0 \text{ for all } i > n\}.$$

   (a) Show that W_n is a subspace of V for all positive integers n, then show that $\bigcup_{n=1}^{\infty} W_n$ is also a subspace of V.

   (b) Find $\dim \left(V \,/\, \bigcup_{n=1}^{\infty} W_n\right)$. Please explain your answer with all the details.

   (c) Prove or disprove: V is isomorphic to $\bigcup_{n=1}^{\infty} W_n$.