MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 10

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-12-02 (Monday) 23:59***.

1. Let T be a linear operator on a finite-dimensional inner product space V.
   (a) If T is an orthogonal projection, prove that $\|T(x)\|^2 \leq \|x\|^2$ for all $x \in V$. Give an example of a projection for which this inequality does not hold. What can be concluded about a projection for which the equality holds for all $x \in V$?
   (b) Suppose that T is a projection such that $\|T(x)\|^2 \leq \|x\|^2$ for all $x \in V$. Prove that T is an orthogonal projection.
2. Let T be a linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = 2f(x) - f'(x)$. Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T.
3. Let T be a linear operator on a vector space V, and let γ be a cycle of generalized eigenvectors that corresponds to the eigenvalue λ. Prove that $\text{span}(\gamma)$ is a T-invariant subspace of V.
4. Let T be a linear operator on a finite-dimensional vector space whose characteristic polynomial splits, and let λ be an eigenvalue of T.
   (a) Suppose that γ is a basis for K_λ consisting of the union of q disjoint cycles of generalized eigenvectors. Prove that $q \leq \dim(E_\lambda)$.
   (b) Let β be a Jordan canonical basis for T, and suppose that $J = [T]_\beta$ has q Jordan blocks with λ in the diagonal positions. Prove that $q \leq \dim(E_\lambda)$.

Exercises

The following are extra recommended exercises not included in the homework.

1. Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ of T to prove the following results.

   (a) If g is a polynomial, then

   $$g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i.$$

   (b) If $T^n = T_0$ for some n, then $T = T_0$.

   (c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

   (d) There exists a normal operator U on V such that $U^2 = T$.

   (e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

   (f) T is a projection if and only if every eigenvalue of T is 1 or 0.

2. Let

   $$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}.$$

   Find a basis for each generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

3. (a) Let A and B be commuting square matrices, i.e. $AB = BA$. Show that the binomial formula can be applied to $(A+B)^n$, i.e.,

   $$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k,$$

   where $\binom{n}{k}$ is the binomial coefficient.

   (b) Let A be the Jordan block

   $$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

   Find A^4 using part (a).

4. Let T be a linear operator on a finite-dimensional vector space V whose characteristic polynomial splits.

   (a) Suppose that β is a basis for V such that β is a disjoint union of cycles of generalized eigenvectors of T. Prove that β is a Jordan canonical basis for V.

   (b) Suppose that β is a Jordan canonical basis for T, and let λ be an eigenvalue of T. Let $\beta' = \beta \cap K_\lambda$. Prove that β' is a basis for K_λ.