MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 9

Problems

Please give reasons for your solutions to the following homework problems.
*Submit your solution in PDF via the Blackboard system before **2024-11-22 (Friday) 23:59**.*

1. Let T and U be self-adjoint linear operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

2. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

   (a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.

   (b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$.

   (c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.

3. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$
$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

   (a) Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

   (b) Prove that $(T + iI)(T - iI)^{-1}$ is unitary.

4. Let W be a finite-dimensional subspace of an inner product space V. Define $U : V \to V$ by $U(v_1 + v_2) = v_1 - v_2$, where $v_1 \in W$ and $v_2 \in W^\perp$. Prove that U is a self-adjoint unitary operator.

5. Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^\perp$.

Exercises

The following are extra recommended exercises not included in the homework.

1. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2i}(T - T^*).$$

   (a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

   (b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.

   (c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

2. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. If T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

3. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. If W is T-invariant, then W is also T^*-invariant.

4. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that

   (a) $U(W) = W$;

   (b) $W^\perp$ is U-invariant.

5. Let U be a linear operator on a finite-dimensional inner product space V. If $\|U(x)\| = \|x\|$ for all x in some orthonormal basis for V, must U be unitary? Justify your answer with a proof or a counterexample.

6. If T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary root; that is, there exists a unitary operator U such that $T = U^2$.