MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 6

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-10-25 (Friday) 23:59****.*

1. Let $T \in \mathcal{L}(P_2(\mathbb{R}))$ be defined by $T(f(x)) = af(0) + f(-1)(x + x^2)$. Prove that T is not diagonalizable for any $a \in \mathbb{R}$.

2. Let $A \in M_{n \times n}(F)$.

   (a) Show that A and A^T have the same characteristic polynomials and eigenvalues.

   (b) Give an example that A and A^T could have different eigenspaces for a given common eigenvalue.

   (c) Prove that for any common eigenvalue λ, $\gamma_A(\lambda) = \gamma_{A^T}(\lambda)$.

   (d) Prove that if A is diagonalizable, then A^T is also diagonalizable.

3. Let A be a $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with corresponding multiplicities $m_1, m_2, .., m_k$. Prove the following statements.

   (a) $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$

   (b) $\det(A) = \prod_{i=1}^{k} \lambda_i^{m_i}$

4. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, ..., \lambda_k$.

   (a) Prove that $\operatorname{span}(\{x \in V : x \text{ is an eigenvector of } T\} = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots E_{\lambda_k}$.

   (b) Hence, prove that $V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots E_{\lambda_k}$ if T is diagonalizable.

5. Let T be a linear operator on a vector space V, and suppose there exist linearly independent non-zero vectors $u, v \in V$ such that $T(u) = 2v$ and $T(v) = 2u$. Prove that 2 and -2 are eigenvalues of T.
   Hint: Construct eigenvectors corresponding to the eigenvalues.

Exercises

The following are extra recommended exercises not included in the homework.

1. Given the matrix
$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$
Is A diagonalizable? If yes, find its eigenspaces, then find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$; if no, explain.

2. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix
   (a) $V = P_3(\mathbb{R})$ and $T(f(x)) = f'(x) + f''(x)$.
   (b) $V = P_2(\mathbb{R})$ and $T(ax^2 + bx + c) = cx^2 + bx + a$.
   (c) $V = \mathbb{R}^3$ and $T(a_1, a_2, a_3) = (a_2, -a_1, 2a_3)$.
   (d) $V = \mathbb{C}^2$ and $T(z, w) = (z + iw, iz + w)$.
   (e) $V = M_{2\times 2}(R)$ and $T(A) = A^T$.

3. Let T be a linear operator on a vector space V, and let c be an eigenvector of T corresponding to the eigenvalue λ.
   (a) Prove that for any positive integer m, x is an eigenvector of T^m corresponding to the eigenvalue λ^m.
   (b) State and prove the analogous result for matrices.

4. Let A be a $n \times n$ matrix with characteristic polynomial
$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$
   (a) Prove that $a_0 = \det(A)$, then deduce A is invertible if and only if $a_0 \neq 0$.
   (b) Prove that $f(t) = (A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t) + q(t)$, where $q(t) \in P_{n-2}(F)$. Hint: Apply mathematical induction to n.
   (c) Show that $\mathrm{tr}(A) = (-1)^{n-1}a_{n-1}$.

5. Suppose that $\lambda_1, ..., \lambda_n \in \mathbb{R}$ are distinct. Prove that $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.