# MATH2048: Honours Linear Algebra II

## 2024/25 Term 1

# Homework 5

## Problems

*Please give reasons for your solutions to the following homework problems.*
*Submit your solution in PDF via the Blackboard system before* ***2024-10-18 (Friday) 23:59***.

1. Define $f \in (\mathbb{R}^2)^*$ by $f(x, y) = 2x + y$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (3x + 2y, x)$.
   (a) Compute $T^*(f)$.
   (b) Let $\beta$ be the standard ordered basis for $\mathbb{R}^2$ and $\beta^* = \{f_1, f_2\}$ be the dual basis. Compute $[T^*]_{\beta^*}$ by expressing $T^*(f_1)$ and $T^*(f_2)$ as linear combinations of $f_1$ and $f_2$.
   (c) What is the relationship between $[T]_\beta$ and $[T^*]_{\beta^*}$?

2. Prove that a function $T : F^n \to F^m$ is linear if and only if there exist $f_1, f_2, ..., f_m \in (F^n)^*$ such that $T(x) = (f_1(x), f_2(x), ..., f_m(x))$ for all $x \in F^n$.

3. Let $V$ an $W$ be finite-dimensional vector spaces over $F$. Let $\psi_1 : V \to V^{**}$ be defined by $\psi_1(v)(f) = f(v)$ for all $f \in V^*$ and $\psi_2 : W \to W^{**}$ be defined by $\psi_2(w)(g) = g(w)$ for all $g \in W^*$. Note that $\psi_1$ and $\psi_2$ are isomorphisms.
   Let $T : V \to W$ be linear, and define $T^{**} = (T^*)^*$. Prove that $\psi_2 T = T^{**}\psi_1$.

4. Given the matrix
$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}.$$
   (a) Find the characteristic polynomial $f_A(x)$, then prove that $f_A(x)$ splits.
   (b) Determine all the eigenvalues of $A$, then find the set of eigenvectors corresponding to $\lambda$ for each eigenvalue $\lambda$ of $A$.
   (c) Show that there exist a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$, then find an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

5. Let $T$ be a linear operator on a vector space $V$ over the field $F$, and let $g(t)$ be a polynomial with coefficients from $F$.
   (a) Prove that if $x$ is an eigenvector of $T$ with corresponding eigenvalue $\lambda$, then $g(T)(x) = g(\lambda)(x)$. That is, $x$ is an eigenvector of $g(T)$ with corresponding eigenvalue g($\lambda$).
   (b) Let $f_T$ be the characteristic polynomial of $T$. Prove that if $T$ is diagonalizable, then $f(T) = T_0$, the zero operator. (We will see that this result does not depend on the diagonalizability of $T$ in later sections.)

## Exercises

*The following are extra recommended exercises not included in the homework.*

1. For each of the following vector spaces $V$ and ordered bases $\beta$, find explicit formulas for vectors of the dual basis $\beta^*$ for $V^*$.

   (a) $V = \mathbb{R}^3$; $\beta = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$

   (b) $V = P_2(\mathbb{R})$; $\beta = \{1, x, x^2\}$

2. Let $V = \mathbb{R}^3$, and define $f_1, f_2, f_3 \in V^*$ as follows:

   $$f_1(x, y, z) = x - 2y, \quad f_2(x, y, z) = x + y + z, \quad f_3(x, y, z) = y - 3z.$$

   Prove that $\{f_1, f_2, f_3\}$ is a basis for $V^*$, and then find a basis for $V$ for which it is a dual basis.

3. Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard bases $\beta$ and $\gamma$. Define $T : V \to W$ by

   $$T(p(x)) = (p(0) - 2p(1), p(0) + p'(0)),$$

   where $p'(x)$ is the derivative of $p(x)$.

   (a) For $f \in W^*$ defined by $f(a, b) = a - 2b$, compute $T^*(f)$.

   (b) Compute $[T]_\beta^\gamma$ and $[T^*]_{\gamma^*}^{\beta^*}$ independently.

4. Let $V = P_n(F)$, and let $c_0, c_1, ..., c_n$ be distinct scalars in $F$.

   (a) For $0 \leq i \leq n$, define $f_i \in V^*$ by $f_i(p(x)) = p(c_i)$. Prove that $\{f_0, f_1, ..., f_n\}$ is a basis for $V^*$.

   (b) Show that there exist unique polynomials $p_0(x), p_1(x), ..., p_n(x)$ such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. (Hint: Lagrange polynomials)

   (c) For any scalars $a_0, a_1, ..., a_n$ (not necessarily distinct), find the polynomial $q(x)$ of degree at most $n$ such that $q(c_i) = a_i$ for $0 \leq i \leq n$ and show that $q(x)$ is unique.

5. Let $T$ be a linear operator on a vector space $V$.

   (a) Suppose $V$ is finite-dimensional. Prove that $T$ is invertible if and only if zero is not an eigenvalue of $T$.

   (b) Suppose $T$ is invertible. Prove that a scalar $\lambda$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.

6. Let $A, B \in M_{n\times n}(\mathbb{C})$.

   (a) Prove that if $B$ is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible. Hint: Examine $\det(A + cB)$.

   (b) Find nonzero $2 \times 2$ matrices $A$ and $B$ such that both $A$ and $A + cB$ are invertible for all $c \in \mathbb{C}$.