MATH2048: Honours Linear Algebra II

2024/25 Term 1

Homework 4

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-10-04 (Friday) 23:59***.

1. Let V be a finite-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

2. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

   Let β and γ be the standard bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively.

   (a) Compute $[U]_\beta^\gamma, [T]_\beta$, and $[UT]_\beta^\gamma$. Then verify the equation $[UT]_\beta^\gamma = [U]_\beta^\gamma \, [T]_\beta$.

   (b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then verify the equation $[U(h(x))]_\gamma = [U]_\beta^\gamma \, [h(x)]_\beta$.

3. Let V be a finite-dimensional vector space, and $T : V \to V$ be linear such that $T^2 = T$.

   (a) Find all possible linear transformations T.

   (b) Suggest an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

4. Let V be a finite-dimensional vector space, and $U, T : V \to V$ be linear.

   (a) Prove or give a counter-example: If both U, T are isomorphism, then UT is an isomorphism.

   (b) Prove or give a counter-example: If UT is an isomorphism, then both U, T are isomorphisms.

Exercises

The following are extra recommended exercises not included in the homework.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard bases for $\mathbb{R}^2$ and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$.
   (a) Compute $[T]_\beta^\gamma$.
   (b) If $\alpha = \{(1, 2), (2, 3)\}$, compute $[T]_\alpha^\gamma$.

2. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, ..., v_n\}$. Define $v_0 = 0$.
   (a) Prove that there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, ..., n$.
   (b) Compute $[T]_\beta$.

3. Which of the following pairs of vector spaces are isomorphic?
   (a) F^3 and $P_3(F)$
   (b) F^4 and $P_3(F)$
   (c) $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$
   (d) $V = \{A \in M_{2\times 2}(\mathbb{R}) : \text{tr}(A) = 0\}$ and $\mathbb{R}^4$

4. Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see the definition in HW 3) having dimension k. Show that there is a basis β of V such that $[T]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

   where A is a $k \times k$ matrix and O is the $(n - k) \times k$ zero matrix.

5. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ (zero transformation) if and only if $R(T) \subseteq N(T)$.

6. Let A be an $n \times n$ matrix.
   (a) Suppose that $A^2 = O$. Prove that A is not invertible.
   (b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix. Could A be invertible? Explain.

7. Let B be an $n \times n$ matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.