# MATH2048: Honours Linear Algebra II
## 2024/25 Term 1

# Homework 3

## Problems

*Please give reasons for your solutions to the following homework problems.*
*Submit your solution in PDF via the Blackboard system before* ***2024-09-27 (Friday) 23:59****.*

1. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear, $T(1,0) = (1,4,5)$ and $T(1,1) = (2,5,3)$. What is $T(2,3)$? Is $T$ one-to-one?

2. Let $V$ and $W$ be vector spaces and $T : V \to W$ be linear.

   (a) Prove that $T$ is one-to-one if and only if $T$ carries linearly independent subsets of $V$ onto linearly independent subset of $W$.

   (b) Suppose that $T$ is one-to-one and that $S$ is a subset of $V$. Prove that $S$ is linearly independent if and only if $T(S)$ is linearly independent.

   (c) Suppose $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for $V$ and $T$ is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for $W$.

3. Let $V$ be a vector space, and let $T : V \to V$ be linear. Let $W$ be a $T$-*invariant* subspace of $V$, which means that for all $w \in W$, we have $T(w) \in W$. Suppose that $V = R(T) \oplus W$.

   (a) Prove that $W \subseteq N(T)$.

   (b) Show that if $V$ is finite-dimensional, then $W = N(T)$.

   (c) Show by example that the conclusion of (b) is not necessarily true if $V$ is not finite-dimensional.

4. Let $V$ be a vector space over a field $F$. Suppose $S$ is a linearly independent subset of $V$ that is not a basis. Using Zorn's Lemma, prove that there exists a basis of $V$ that contains $S$.

# Exercises

*The following are extra recommended exercises not included in the homework.*

1. Given the matrix
$$A = \begin{pmatrix} 2 & 3 & 5 & 1 \\ 4 & -2 & 3 & -1 \\ 10 & -9 & 4 & -4 \end{pmatrix}.$$
Let $L_A : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $L_A(x) = Ax$ for all $x \in \mathbb{R}^4$.
   (a) Find $R(L_A)$ and $N(L_A)$.
   (b) Verify $\dim(V) = \dim(R(L_A)) + \dim(N(L_A))$.

2. Let $T : \mathbb{C} \to \mathbb{C}$ be a function defined by $T(z) = \overline{z}$. Prove that $T$ is not linear.

3. Find all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

4. Let $V$ and $W$ be vector spaces with subspaces $V_1$ and $W_1$ respectively. If $T : V \to W$ is linear, prove that $T(v_1)$ is a subspace of $W$ and that $\{x \in V : T(x) \in W_1\}$ is a subspace of $V$.

5. Let $V$ be a finite-dimensional vector space and $T : V \to V$ be linear.
   (a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.
   (b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.
   (c) Give an example of $V$ and $T$ such that $R(T) \oplus N(T)$.

   Be careful to say in parts (a) and (b) where finite-dimensionality is used.