MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 2

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-09-20 (Friday) 23:59****.*

1. Let V and W be vector spaces over the field F, and let $S_1 = \{v_1, ..., v_n\}$ and $S_2 = \{w_1, ..., w_n\}$ with $n > 1$ be sets of vectors in V and W respectively. Consider the direct product $V \times W$.

   (a) Prove or disprove: If S_1 and S_2 are linearly independent in U and V respectively, then $\{(v_1, w_1), ..., (v_n, w_n)\}$ is linearly independent in $V \times W$.

   (b) Prove the following statement in the general case or give a counterexample with some specific V and W: If $\{(v_1, w_1), ..., (v_n, w_n)\}$ is linearly independent in $V \times W$, then S_1 and S_2 are linearly independent in U and V respectively.

2. Let V be a vector space with subspace W. Consider the set $V/W = \{v + W : v \in V\}$.

   (a) Prove that $v + W$ is a subspace if and only if $v \in W$.

   (b) Prove that addition and scalar multiplication are well-defined in V/W. That is, if $v + W = v' + W$, then for any $v'' + W \in V/W$, we have

   $$(v + W) + (v'' + W) = (v' + W) + (v'' + W)$$
   $$a \cdot (v + W) = a \cdot (v' + W).$$

   (c) Hence, prove that V/W is a vector space over F with the addition and scalar multiplication defined above.

   (d) Prove or disprove: If $v_1 + W$ and $v_2 + W$ is linearly independent in V/W, then v_1 and v_2 are linearly independent in V.

3. Suppose $W = \{(x, y, x, y) \in F^4 \mid x, y \in F\}$, where F is a field.

   (a) Prove that W is a subspace of $V = F^4$. What is the dimension of W?

   (b) Suggest a basis for V/W.

4. Let V be the space of all functions from $\mathbb{N}$ to $\mathbb{Z}_2$ that are zero outside a finite subset of $\mathbb{N}$. Suggest a basis for V. Suppose we do not allow infinite sum. Then, could you find a countable basis for $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$, which contains all functions from $\mathbb{N}$ to $\mathbb{Z}_2$? Please explain. (Remark: $\mathbb{Z}_2$ is the field with two elements 0 and 1.)

Exercises

The following are extra recommended exercises not included in the homework.

1. What are the relationships between projection operators, direct sums, direct products, and quotient spaces?

2. Given the matrix
$$A = \begin{pmatrix} 3 & 5 & 1 \\ 6 & 10 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

   (a) Find the null space $N(A)$.

   (b) Find the quotient space $\mathbb{R}^3 / N(A)$.

3. Suppose that U is a subspace of V such that V/U is finite-dimensional.

   (a) Show that if W is finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.

   (b) Prove that there exists a finite-dimensional subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

4. Let $V = \mathbb{C}^3$ and $W = M_{2\times 2}(\mathbb{C})$ be vector spaces over $\mathbb{R}$. Is $V \times W$ well-defined? If yes, state its dimension and suggest a basis for $V \times W$.

5. Let W_1 be the space of $n \times n$ matrices over $\mathbb{R}$ whose trace is zero. The trace of a square matrix is defined as the sum of its diagonal entries. Find a subspace W_2 such that $\mathbb{R}^{n\times n} = W_1 \oplus W_2$. (Hint: consider the nature of matrices in W_1 and what kind of matrices are not in W_1.)

6. Let U be a subspace of $P_n(\mathbb{R})$ such that $f(0) = 0$ for all $f \in U$.

   (a) Show that U is a subspace of $P_n(\mathbb{R})$.

   (b) Describe the quotient space $P_n(\mathbb{R})/U$.