MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 1

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-09-13 (Friday) 23:59****.*

1. Show that the set of differentiable real-valued functions f on $\mathbb{R}$ such that $f'(0) = 2f(1)$ is a vector space.

2. Let V be a vector space over an infinite field F.

   (a) Let W_1, W_2 be subspaces of V such that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Prove that $W_1 \cup W_2$ is not a subspace of V.

   (b) Construct a nontrivial vector space V and a set of subspaces $\{W_i\}_{i=0}^{\infty}$ of V such that $W_i \nsubseteq W_j$ for all $i \neq j$ and $\bigcup_{i=0}^{\infty} W_i$ is a subspace of V.
   Hint: Consider $V = \mathbb{Q}^2$, and W_i are the set $\{(0, q) : q \in \mathbb{Q}\}$ and the sets $\{(q, pq) : q \in \mathbb{Q}\}$ for all $p \in \mathbb{Q}$.

3. Suppose $v_1, ..., v_n$ is linearly independent in V. For any nonzero $a_1, ..., a_n \in F$, Prove that the list

$$a_1v_1 + a_2v_2, a_2v_2 + a_3v_3, ..., a_{n-1}v_{n-1} + a_nv_n$$

   is linearly independent.

4. Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

   Hint: Start with a basis $\{u_1, u_2, ..., u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, ..., u_k, v_1, v_2, ..., v_m\}$ for W_1 and to a basis $\{u_1, u_2, ..., u_k, w_1, w_2, ..., w_p\}$ for W_2.

5. Let $V = M_{n\times n}(\mathbb{C})$ be a vector space over $\mathbb{R}$. Given that the sets

$$U = \{A \in V : \text{all entries of } A \text{ are real}\}$$
$$W = \{A \in V : \text{all entries of } A \text{ are purely imaginary}\}$$

   are subspaces of V (no need to prove this).
   Show that $V = U \oplus W$. What is the dimension of V?

Exercises

The following are extra recommended exercises not included in the homework.

1. Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) = (a_1, a_2).$$

   Is V a vector space over F with these operations? Justify your answers.

2. Consider the vector space of all 2×2 matrices with real entries. Is the set of all 2×2 diagonal matrices a subspace of this vector space? Is the set of all 2×2 upper triangular matrices a subspace of this vector space?

3. Find a basis for the null space of the linear system represented by the following matrix:

$$\begin{bmatrix} -2 & 0 & 3 & 1 \\ 5 & 9 & -6 & 5 \\ 1 & 3 & -1 & 2 \end{bmatrix}$$

4. (a) Are the vectors $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ linearly independent?

   (b) Are the vectors $\begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ linearly independent?

5. Consider the following set of vectors: $(i + 1, i), (2, i + 1)$.

   (a) Are these vectors linearly independent in $\mathbb{C}^2$ over the field $\mathbb{C}$?

   (b) Are these vectors linearly independent in $\mathbb{C}^2$ over the field $\mathbb{R}$?

6. Given that the list $(t - 6, 3, 1), (1, -1, 2), (7, 3, t)$ is linearly dependent in $\mathbb{R}^3$. Find t.

7. Suppose $p_0, p_1, ..., p_n \in \mathcal{P}_n(F)$ are polynomials such that $p_k(2) = 0$ for $k \in \{0, ..., n\}$. Prove that $p_0, p_1, ..., p_n$ is not linearly independent.

8. Suppose $U = \{(x, y, x + y, x - y, 2x) \in F^5 \mid x, y \in F\}$, where F is a field.

   (a) Prove that U is a vector space. What is the dimension of U?

   (b) Find a subspace $W \in F^5$ such that $U \oplus W = F^5$.

9. A function is called *periodic* if there exists $p \in \mathbb{R}^+$ such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?