MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 4

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-10-04 (Friday) 23:59****.*

1. Let V be a finite-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

   Proof. This follows from basic properties of matrix representation.

2. Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

   Let β and γ be the standard bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively.

   (a) Compute $[U]_\beta^\gamma, [T]_\beta$, and $[UT]_\beta^\gamma$. Then verify the equation $[UT]_\beta^\gamma = [U]_\beta^\gamma \, [T]_\beta$.
   Proof. We compute $T(1) = 2$, $T(x) = 3 + 3x$, $T(x^2) = 4x^2 + 6x$, $U(1) = (1, 0, 1)$, $U(x) = (1, 0, -1)$, and $U(x^2) = (0, 1, 0)$. The remaining part is just basic matrix multiplication. $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$, $[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$, $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

   (b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then verify the equation $[U(h(x))]_\gamma = [U]_\beta^\gamma \, [h(x)]_\beta$.
   Proof. $[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

3. Let V be a finite-dimensional vector space, and $T : V \to V$ be linear such that $T^2 = T$.

   (a) Find all possible linear transformations T.

   Proof. We characterize all such T by establishing a one-one correspondence between them and the set of all the subspaces of V via the operator f that maps T to $R(T)$. For any such T, notice that $T(T(x)) = T(x)$ for any $x \in V$. This implies $R(T)$ is a T-invariant subspace. By *Problem 3* of the previous homework, $V = R(T) \oplus N(T)$. So, any T_1 and T_2 sharing the same range must share the same kernel. Further, let $y \in R(T_1) = R(T_2)$ and suppose $y = T_1(x_1) = T_2(x_2)$. We see $T_1(y) = T_1(T_1(x_1)) = T_1(x_1) = y = T_2(x_2) = T_2(T_2(x_2)) = T_2(y)$. This proves $T_1 = T_2$ and hence the injectivity of the operator f. For the surjectivity of f, let

$W \subset V$ be an arbitrary subspace with basis $\beta = \{v_1, \cdots, v_k\}$. We extend β to a basis of V: $\{v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+n}\}$. Define T by letting $T(v_i) = v_i$ for $i = 1, \cdots, k$ and $T(v_{i+j}) = 0$ for $j = 1, \cdots, n$. Clearly, T fulfills the given requirements.

(b) Suggest an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Proof. This follows from the surjectivity part of the previous proof. We observe that each T is a projection onto its range.

4. Let V be a finite-dimensional vector space, and $U, T : V \to V$ be linear.

(a) Prove or give a counter-example: If both U, T are isomorphism, then UT is an isomorphism.

Proof. Recall from MATH1050 that UT is bijective if U and T are both bijective. It remains to prove UT is linear. Indeed, for any $x, y \in V$, we have $UT(cx + y) = U(cT(x) + T(y)) = cUT(x) + UT(y)$.

(b) Prove or give a counter-example: If UT is an isomorphism, then both U, T are isomorphisms.

Proof. It suffices to show injectivity by the rank-nullity theorem. Take arbitrary $x \in ker(T)$, we see $UT(x) = U(0) = 0$. So, x lies in $ker(UT)$ and must be zero. This shows that T is isomorphic. Now, take arbitrary $y \in ker(U)$. Since T is already isomorphic, $y = T(x)$ for some x. So, $U(y) = UT(x) = 0$. This forces x to be 0 and hence $y = 0$. So, U is also isomorphic.