MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 3

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-09-27 (Friday) 23:59***.

1. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear, $T(1,0) = (1,4,5)$ and $T(1,1) = (2,5,3)$. What is $T(2,3)$? Is T one-to-one?

   Proof. $T(2,3) = T(-1,0) + 3T(1,1) = (6,15,9) + (-1,-4,-5) = (5,11,4)$. T is one-to-one from $\mathbb{R}^2$ to its range.

2. Let V and W be vector spaces and $T : V \to W$ be linear.

   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subset of W.

   Proof. If T is one-to-one, the kernel of T only contains 0. Let $\{v_1, \cdots, v_n\}$ be a linearly independent subset of V. Then, $c_1T(v_1) + \cdots + c_nT(v_n) = T(c_1v_1 + \cdots + c_nv_n) = 0$ implies $c_1 = \cdots = c_n = 0$. This proves the linearly independence of $\{T(v_1), \cdots, T(v_n)\}$
   Conversely, suppose $\{T(v_1), \cdots, T(v_n)\}$ is linearly independent. $c_1v_1+\cdots+c_nv_n = 0$ for some $c_1, \cdots, c_n$ implies $T(c_1v_1+\cdots+c_nv_n) = c_1T(v_1)+\cdots+c_nT(v_n) = 0$. Hence, $c_1 = \cdots = c_n = 0$. The proof is done.

   (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

   Proof. This directly follows from the previous argument.

   (c) Suppose $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.

   Proof. Since T is one-to-one, $\{T(v_1), \cdots, T(v_n)\}$ is linearly independent in W. Since the linear map T is also onto, $\{T(v_1), \cdots, T(v_n)\}$ spans W. Hence, $T(\beta)$ is a basis for W.

3. Let V be a vector space, and let $T : V \to V$ be linear. Let W be a T-*invariant* subspace of V, which means that for all $w \in W$, we have $T(w) \in W$. Suppose that $V = R(T) \oplus W$.

   (a) Prove that $W \subseteq N(T)$.

   Proof. If some $w \in W$ satisfies $T(w) = v \neq 0$. By the T-invariance of W, v also lies in W. This contradicts the fact that $R(T) \cap W = \{0\}$

(b) Show that if V is finite-dimensional, then $W = N(T)$.

Proof. Suppose the dimension of V is finite. By the rank-nullity theorem, $dimV = dim(R(T)) + dim(N(T))$. Since $V = R(T) \oplus W$, we also have $dimV = dim(R(T)) + dimW$. So, $dimW = dim(N(T))$. In the previous result, we have proved $W \subseteq N(T)$. We instantly see $W = N(T)$.

(c) Show by example that the conclusion of (b) is not necessarily true if V is not finite-dimensional.

Proof. Take V as the space of real-valued smooth functions, T as the differentiation, and W as $\{0\}$. $R(T)$ is still the whole space and $N(T)$ contains all constant functions. $V = R(T) \oplus W$ but $W \subseteq N(T)$.

4. Let V be a vector space over a field F. Suppose S is a linearly independent subset of V that is not a basis. Using Zorn's Lemma, prove that there exists a basis of V that contains S.

Proof. Let U be a non-zero subspace of an infinite dimensional vector space V over a field F. Let $L \subset U$ be a basis for U. We aim to show that L can be extended to a basis of V using Zorn's lemma. To do so, consider the set S of all linearly independent subsets of V that contain L. This set S is non-empty since it contains L, and it is partially ordered by inclusion. Given a chain C in S (a totally ordered subset of S), we can show that it has an upper bound in S. Take $L' = C$. As each set in C is linearly independent and each set in the chain contains L, L' is linearly independent and contains L, so $L' \subset S$. Therefore, L' is an upper bound for the chain C in S. By Zorn's lemma, S has a maximal element, say B. If B is not a basis for V, then it must not span V. There exists a vector $v \in V$ not in the span of B. Add v to B, to get a larger linearly independent set contradicting the maximality of B. Hence, B is a basis for V. Finally, let $W = B \setminus L$. Then W is a subspace of V that is disjoint from U (since L is a basis for U), and $V = U \oplus W$. This is because any vector $vinV$ can be written uniquely as $v = u + w$ for some $uinU$ and $winW$. Here, the uniqueness follows from the fact that $B = L \cup W$ is a basis for V.