MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 2 Solutions

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-09-20 (Friday) 23:59***.

1. Let V and W be vector spaces over the field F, and let $S_1 = \{v_1, ..., v_n\}$ and $S_2 = \{w_1, ..., w_n\}$ with $n > 1$ be sets of vectors in V and W respectively. Consider the direct product $V \times W$.

   (a) Prove or disprove: If S_1 and S_2 are linearly independent in V and W respectively, then $\{(v_1, w_1), ..., (v_n, w_n)\}$ is linearly independent in $V \times W$.

   Proof. Suppose $a_1(v_1, w_1) + \cdots + a_n(v_n, w_n) = (a_1v_1 + \cdots + a_nv_n, a_1w_1 + \cdots + a_nw_n) = (0, 0)$. Clearly, $a_1 = \cdots = a_n = 0$.

   (b) Prove the following statement in the general case or give a counterexample with some specific V and W: If $\{(v_1, w_1), ..., (v_n, w_n)\}$ is linearly independent in $V \times W$, then S_1 and S_2 are linearly independent in U and V respectively.

   Proof. Consider the counterexample $V = W = \mathbb{R}^2$ and $v_1 = (1, 0), v_2 = (1, 0), w_1 = (1, 0), w_2 = (0, 1)$. Indeed, $\{(v_1, w_1), (v_2, w_2)\} = \{(1, 0, 1, 0), (1, 0, 0, 1)\}$ is linearly independent in $\mathbb{R}^4$ but S_1 is not in $\mathbb{R}^2$.

2. Let V be a vector space with subspace W. Consider the set $V/W = \{v + W : v \in V\}$.

   (a) Prove that $v + W$ is a subspace if and only if $v \in W$.

   Proof. If $v + W$ is a subspace, we have $0 \in v + W$. This implies $-v \in W$ and hence $v \in W$. Conversely, if $v \in W$, $v + W$ is a subspace because $v + W = W$.

   (b) Prove that addition and scalar multiplication are well-defined in V/W. That is, if $v + W = v' + W$, then for any $v'' + W \in V/W$, we have

   $$(v + W) + (v'' + W) = (v' + W) + (v'' + W)$$
   $$a \cdot (v + W) = a \cdot (v' + W).$$

   Proof. Recall the definition $(v + W) + (v'' + W) = (v + v'') + W$. Take arbitrary $(v + v'' + w) \in (v + v'') + W$. The condition $v + W = v' + W$ implies $v + w = v' + w_1$ for some $w_1 \in W$. So, $v + v'' + w = v'' + v' + w_1 \in (v' + v'') + W$. For the scalar multiplication, also recall first the definition $a \cdot (v + W) = av + W$. Take arbitrary $w \in W$, we see $v + w/a = v' + w_1/a$ for some w_1. So, $av + w = av' + w_1 \in a \cdot (v' + W)$.

(c) Hence, prove that V/W is a vector space over F with the addition and scalar multiplication defined above.

Proof. This is a direct consequence of the previous question and the definition of vector spaces.

(d) Prove or disprove: If $v_1 + W$ and $v_2 + W$ is linearly independent in V/W, then v_1 and v_2 are linearly independent in V.

Proof. Suppose $a_1 v_1 + a_2 v_2 = 0$. Then, $0 + W = (a_1 v_1 + a_2 v_2)W = a_1(v_1 + W) + a_2(v_2 + W)$. By the assumption that $v_1 + W$ and $v_2 + W$ is linearly independent in V/W, we have $a_1 = a_2 = 0$. Hence, v_1 and v_2 are linearly independent in V.

3. Suppose $W = \{(x, y, x, y) \in F^4 \mid x, y \in F\}$, where F is a field.

(a) Prove that W is a subspace of $V = F^4$. What is the dimension of W?

Proof. This directly follows from the definition. The dimension of W is 2.

(b) Suggest a basis for V/W.

Proof. Observe that a basis for W is $\{(1, 0, 1, 0), (0, 1, 0, 1)\}$. We extend it to a basis of V: $\{(1, 0, 1, 0), (0, 1, 0, 1), (1, 0, -1, 0), (0, 1, 0, -1)\}$. So, a basis for V/W is $\{(1, 0, -1, 0) + W, (0, 1, 0, -1) + W\}$.

4. Let V be the space of all functions from $\mathbb{N}$ to $\mathbb{Z}_2$ that are zero outside a finite subset of $\mathbb{N}$. Suggest a basis for V. Suppose we do not allow infinite sum. Then, could you find a countable basis for $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$, which contains all functions from $\mathbb{N}$ to $\mathbb{Z}_2$? Please explain. (Remark: $\mathbb{Z}_2$ is the field with two elements 0 and 1.)

Proof. For $i \geq 1$, let $f_i(k) = 1$ for $k = i$ and 0 otherwise. $(f_i)_{i\geq 1}$ is a basis for V. Now we prove by contradiction there is no countable basis for $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$. First recall from MATH1050 two important facts: (i)$\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$ has cardinality $2^{\mathbb{N}}$ and is therefore uncountable; (ii)the countable union of countable sets is countable. Suppose $(f_i)_{i\geq 1}$ is a countable basis for $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$. Let $\mathcal{F}_k$ be the subspace generated by $\{f_1, \cdots, f_k\}$. $\mathcal{F}_k$ consists of functions of the type $a_1 f_1 + \cdots + a_k f_k$, where $a_i \in 0, 1$. There is a nature correspondence between $\mathcal{F}_k$ and $\mathbb{Z}_2^k$ by sending $\{f_1, \cdots, f_k\}$ to $(a_1, \cdots, a_k)$. As $\mathbb{Z}_2^k$ is countable, $\mathcal{F}_k$ is therefore countable for any k. Notice that $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2) = \cup_{k\geq 1}\mathcal{F}_k$. We instantly have $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$ is countable, which is a contradiction.