MATH2048: Honours Linear Algebra II
2024/25 Term 1

Homework 1 Sol

Problems

Please give reasons for your solutions to the following homework problems.
Submit your solution in PDF via the Blackboard system before ***2024-09-13 (Friday) 23:59***.

1. Show that the set of differentiable real-valued functions f on $\mathbb{R}$ such that $f'(0) = 2f(1)$ is a vector space.

   Proof. It suffices to check that it is a subspace of the space of real-valued functions. Let $a \in \mathbb{R}$, f, g be arbitrary. Then $(af+g)'(0) = af'(0)+g'(0) = 2af(1)+2g(1) = 2(af+g)(1)$. The result follows.

2. Let V be a vector space over an infinite field F.

   (a) Let W_1, W_2 be subspaces of V such that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Prove that $W_1 \cup W_2$ is not a subspace of V.

   Proof. We prove it by contradiction. Assume first $W_1 \cup W_2$ is a subspace. Find nontrivial $x \in W_1$ and $y \in W_2$ such that $x \notin W_2$ and $y \notin W_1$. If $(x+y) \in W_1$. Then, $(x+y)+(-x) = y \in W_1$, a contradiction. If $(x+y) \in W_2$. Then, $(x+y)+(-y) = x \in W_2$, also a contradiction.

   (b) Construct a nontrivial vector space V and a set of subspaces $\{W_i\}_{i=0}^{\infty}$ of V such that $W_i \not\subseteq W_j$ for all $i \neq j$ and $\bigcup_{i=0}^{\infty} W_i$ is a subspace of V.
   Hint: Consider $V = \mathbb{Q}^2$, and W_i are the set $\{(0, q) : q \in \mathbb{Q}\}$ and the sets $\{(q, pq) : q \in \mathbb{Q}\}$ for all $p \in \mathbb{Q}$.

   Proof. Following the hint, let $p_i, i \geq 1$ be an enumeration of rational numbers. Let $W_0 = span\{(0,1)\}$, and $W_i = span\{(1, p_i)\}$. Notice that $(1, p_i) = (q, qp_j)$ for some q implies $q = 1$ and $p_i = p_j$. Thus, the subspaces W_i, $i \geq 0$ satisfy the given condition. Besides, since $(q_1, q_2) = q_1(1, q_2/q_1)$ for any pair (q_1, q_2), we have $\mathbb{Q}^2 = \cup_i W_i$. This finishes the proof.

3. Suppose $v_1, ..., v_n$ is linearly independent in V. For any nonzero $a_1, ..., a_n \in F$, Prove that the list

   $$a_1v_1 + a_2v_2, a_2v_2 + a_3v_3, ..., a_{n-1}v_{n-1} + a_nv_n$$

   is linearly independent.

   Proof. Suppose $b_1(a_1v_1 + a_2v_2) + \cdots + b_{n-1}(a_{n-1}v_{n-1} + a_nv_n) = 0$. Simple algebra yields $a_1b_1v_1 + (a_2b_1 + b_2a_2)v_2 + \cdots + a_nb_{n-1}v_n = 0$. Since $v_1, \cdots, v_n$ is linearly independent in V, all of the coefficients should zero. a_1 is nonzero instantly gives $b_1 = 0$. By an induction argument, we get b_i are all zero for $i = 1, \cdots, n-1$. Hence, the given list is linearly independent.

4. Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Hint: Start with a basis $\{u_1, u_2, ..., u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, ..., u_k, v_1, v_2, ..., v_m\}$ for W_1 and to a basis $\{u_1, u_2, ..., u_k, w_1, w_2, ..., w_p\}$ for W_2.

Proof. Following the hint, notice that all the elements in $W_1 \cap W_2$ can be written as a linear combination of $\{u_1, u_2, \cdots, u_k\}$. So, all of $\{v_1, \cdots, v_m\}$ are not in W_2 and all of $\{w_1, \cdots, w_p\}$ are not in W_1. What follows is that $\{u_1, \cdots, u_k, v_1, \cdots, v_m, w_1, \cdots, w_p\}$ is a basis for $W_1 + W_2$. Hence, $\dim(W_1 + W_2) = k + m + p = (k + m) + (k + p) - k = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

5. Let $V = M_{n\times n}(\mathbb{C})$ be a vector space over $\mathbb{R}$. Given that the sets

$$U = \{A \in V : \text{all entries of } A \text{ are real}\}$$
$$W = \{A \in V : \text{all entries of } A \text{ are purely imaginary}\}$$

are subspaces of V (no need to prove this).
Show that $V = U \oplus W$. What is the dimension of V?

Proof. Let $A \in V$ and denote by a_{ij} the (i, j)-th entry of A. Write $a_{ij} = b_{ij} + ic_{ij}$, where b_{ij} and c_{ij} is real. Take $B \in U$ and $C \in W$ such that the (i, j)-th entry of B is b_{ij} and the (i, j)-th entry of C is ic_{ij}. Clearly, $A = B + C$. This proves $V = U + W$. Besides, notice that $U \cap W$ contains only the zero matrix. This desired result then follows. Using the formula proved in the previous problem, the dimension of V is $2n^2$.