MATH2040A Week 12 Tutorial Notes

In this tutorial note, I will also denote the complex conjugate $\overline{z}$ of a complex number z as z^*.

1 Adjoint Operator

On an inner product space V, a map $T^* : V \to V$ is an *adjoint operator* of $T \in L(V)$ if $\langle\, Tx,\, y \,\rangle = \langle\, x,\, T^*y \,\rangle$ for all $x, y \in V$. As covered in lecture,

- if T^* exists, it is unique and linear, and $(T^*)^* = T$
- if T^*, U^* exist, then $(TU)^* = U^*T^*$
- if T is invertible and $T^*, (T^{-1})^*$ exist, then $(T^*)^{-1} = (T^{-1})^*$
- (textbook Sec. 6.4 Q7) if $W \subseteq V$ is a T-invariant subspace, then $W^\perp$ is T^*-invariant

and if V is *finite dimensional*,

- *every* linear map $T \in L(V)$ has an/the adjoint
- if α is an orthonormal basis, $[T^*]_\alpha = ([T]_\alpha)^*$, the conjugate transpose of the matrix $[T]_\alpha$
- $* : L(V) \to L(V)$ is *conjugate* linear: $(aT + bU)^* = a^*T^* + b^*U^*$

Adjoint of $T \in L(V, W)$ can be defined in the same way, in which case $T^* \in L(W, V)$ (if it exists)[1].

The existence of adjoint operator in *finite dimensional* spaces relies on the following *important* theorem:

Theorem 1.1 (Riesz Representation Theorem)**.** *If V is a finite dimensional inner product space, then for all $f \in V^* = L(V, F)$, there exists a unique $v_f \in V$ such that $f(v) = \langle\, v,\, v_f \,\rangle$ for all $v \in V$.*

That is, for *finite dimensional* inner product space, every linear functional is induced by some vector via the inner product pairing, and this gives a *conjugate linear* isomorphism between V^* and V.

Here are some properties that are handy:

- if λ is an eigenvalue of T, then λ^* is an eigenvalue of T^*
- (textbook Sec. 6.3 Q12) $\mathsf{R}(\, T^* \,)^\perp = \mathsf{N}(\, T \,)$, $\mathsf{R}(\, T \,)^\perp = \mathsf{N}(\, T^* \,)$.
  If V is *finite dimensional*, $\mathsf{R}(\, T^* \,) = \mathsf{N}(\, T \,)^\perp$, $\mathsf{R}(\, T \,) = \mathsf{N}(\, T^* \,)^\perp$
- (textbook Sec. 6.3 Q13) $\mathsf{N}(\, T^*T \,) = \mathsf{N}(\, T \,)$. If V is *finite dimensional*, $\operatorname{rank}(T^*) = \operatorname{rank}(T)$
  (that is, *row rank equals column rank*)

1.1 Computation of Adjoint Operator

Given a $T \in L(V)$ on a finite dimensional inner product space V, how to find T^*? One approach is to

1. take an orthonormal basis $\alpha = \{\, e_1, \ldots, e_n \,\}$
2. compute $[T^*]_\alpha = ([T]_\alpha)^*$
3. revert back from matrix representation $[T^*]_\alpha$ to a linear map T^* on V

Of course, there are many approaches to do this, and many ways to simplify these computations.

[1] See textbook Sec. 6.3 Q15-17.

2 Normal and Self-adjoint Maps

For a linear map $T \in L(V)$ with adjoint T^*, T is *normal* if $TT^* = T^*T$, *self-adjoint* if $T^* = T$.
The basic properties of normal operators are

- $\| T(v) \| = \| T^*(v) \|$ for all v
- if V is a complex space, $T_1 = \frac{1}{2}(T + T^*)$, $T_2 = \frac{1}{2i}(T - T^*)$, then $T_1T_2 = T_2T_1$
- if $Tv = \lambda v$, then $T^*v = \lambda^* v$. That is, $E_{\lambda^*}(T^*) = E_\lambda(T)$
- if v_1, v_2 are eigenvectors with distinct eigenvalues, then $v_1 \perp v_2$

The basic properties of self-adjoint operators are

- every eigenvalue of T must be real
- if V is finite dimensional, characteristic polynomial of T splits. In particular, T has an eigenvalue
- (textbook Sec. 6.4 Q11) $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$. This is in fact equivalent to T being self-adjoint.

Theorem 2.1 (Schur Decomposition)**.** *Let V be a* finite dimensional *inner product space, $T \in L(V)$ with a characteristic polynomial that splits. Then there exists an orthonormal basis α of V such that $[T]_\alpha$ is upper triangular.*

Theorem 2.2 (Spectral Theorem)**.** *Let V be a* finite dimensional *inner product space, $T \in L(V)$.*

- *Suppose V is a* complex *vector space. Then T has an orthonormal eigenbasis if and only if T is* normal*.*
- *Suppose V is a* real *vector space. Then T has an orthonormal eigenbasis if and only if T is* self-adjoint*.*

In another word, T is normal (on complex space) / self-adjoint (on real space)[2] if and only if there exists an orthonormal basis α such that $[T]_\alpha$ is diagonal.

2.1 Computation of Basis for Spectral Theorem

It is easy to find such an orthonormal eigenbasis:

1. Diagonalize T, as done in previous lectures.
   You now have all eigenvalues $\lambda_1, \dots, \lambda_n$ of T and a basis β_j for each eigenspace.
2. Apply Gram–Schmidt process on each of β_j for orthonormal basis α_j for each of the eigenspaces
3. Union all such orthonormal bases to get an orthonormal eigenbasis $\alpha = \bigcup \alpha_j$

Computationally, this is the same as diagonalizing a lienar map with some extra works (for computing Gram–Schmidt process).

3 Exercises

1. Let V be a finite dimensional inner product space with an ordered basis $\beta = \{ v_1, \dots, v_n \}$ *not necessarily orthonormal*, and $T \in L(V)$.
   Let $G \in F^{n \times n}$ be the Gram matrix of β defined by $G_{jk} = \langle v_k, v_j \rangle$ for each j, k.
   Find $[T^*]_\beta$ with $[T]_\beta$ and the Gram matrix G of β.

[2] For real *normal* linear map, an analog result exists, although the proof (generally) requires *complexification* of the space. See Axler Sect. 9.B.

Solution: Let $\alpha = \{ e_1, \ldots, e_n \}$ be an orthonormal basis of V.
Since α is orthonormal, $[T^*]_\alpha = [T]_\alpha^*$, so on $R = [\mathrm{Id}]_\beta^\alpha$, R is invertible and $R_{jk} = \langle v_k, e_j \rangle$, so

$$(R^*R)_{jk} = \sum_l \overline{\langle v_j, e_l \rangle} \langle v_k, e_l \rangle = \left\langle v_k, \sum_l \langle v_j, e_l \rangle e_l \right\rangle = \langle v_k, v_j \rangle = G_{jk}$$

and thus

$$\begin{aligned}
[T^*]_\beta &= [\mathrm{Id}]_\alpha^\beta [T^*]_\alpha [\mathrm{Id}]_\beta^\alpha \\
&= [\mathrm{Id}]_\alpha^\beta ([T]_\alpha)^* [\mathrm{Id}]_\beta^\alpha \\
&= [\mathrm{Id}]_\alpha^\beta ([\mathrm{Id}]_\beta^\alpha [T]_\beta [\mathrm{Id}]_\alpha^\beta)^* [\mathrm{Id}]_\beta^\alpha \\
&= [\mathrm{Id}]_\alpha^\beta ([\mathrm{Id}]_\alpha^\beta)^* ([T]_\beta)^* ([\mathrm{Id}]_\beta^\alpha)^* [\mathrm{Id}]_\beta^\alpha \\
&= R^{-1}(R^{-1})^*([T]_\beta)^* R^* R \\
&= (R^*R)^{-1}([T]_\beta)^* R^* R \\
&= G^{-1}([T]_\beta)^* G
\end{aligned}$$

Note

See also the note for tutorial 10 exercise 4.

2. Let V be a finite dimensional complex inner product space, and $T \in L(V)$ be normal.
   Let $\lambda \in \mathbb{C}$, $v \in V$ such that $\| v \| = 1$ and $\| Tv - \lambda v \| < \epsilon$ for some $\epsilon > 0$.
   Show that $| \lambda - \lambda' | < \epsilon$ for some eigenvalue $\lambda' \in \mathbb{C}$ of T.

Solution: By spectral theorem, there exists an orthonormal eigenbasis $\{ e_1, \ldots, e_n \}$ of T for V. Let the associated eigenvalues be $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.
Then $v = \sum \langle v, e_j \rangle e_j$, so $1 = \| v \|^2 = \sum | \langle v, e_j \rangle |^2$ and $Tv = \sum \langle v, e_j \rangle \lambda_j e_j$.
Suppose on the contrary that $| \lambda - \lambda_j | \geq \epsilon$ for each j. Then we have

$$\epsilon^2 > \| Tv - \lambda v \|^2 = \left\| \sum \langle v, e_j \rangle (\lambda_j - \lambda) e_j \right\|^2 = \sum | \langle v, e_j \rangle |^2 | \lambda_j - \lambda |^2 \geq \sum | \langle v, e_j \rangle |^2 \epsilon^2 = \epsilon^2$$

Contradiction arises. So $| \lambda - \lambda_j | < \epsilon$ for some j.

Note

We can also show that there exists $v' \in E_{\lambda'}$ such that $\| v - v' \| \leq C\epsilon$ with some $C > 0$ depending only on (the distribution of the eigenvalues of) T.

3. Let V be an inner product space, and $T \in L(V)$ with adjoint T^*.
   Suppose $T^*T = T^2$. Show that T is self-adjoint.

Solution: By direct computation, for $v \in V$,

$$\begin{aligned}
\| (T - T^*)v \|^2 &= \langle (T - T^*)v, (T - T^*)v \rangle \\
&= \langle (T^* - T)(T - T^*)v, v \rangle \\
&= \left\langle (T^*T - T^2 - (T^*)^2 + TT^*)v, v \right\rangle \\
&= \left\langle (TT^* - (T^*)^2)v, v \right\rangle
\end{aligned}$$

so to show that $T^* = T$, it suffices to show that $TT^* = (T^*)^2$, or equivalently $TT^* = (TT^*)^* = ((T^*)^2)^* = T^2$, that is $T(T - T^*) = 0$.

By assumption, $T^*T = T^2$, so $T^*T = (T^*T)^* = (T^2)^* = (T^*)^2$, that is $T^*(T - T^*) = 0$.
Let $v \in V$. Then on $w = (T - T^*)v$, $T^*w = T^*(T - T^*)v = (T^*T - (T^*)^2)v = 0$.
This implies $\| Tw \|^2 = \langle Tw, Tw \rangle = \langle T^*Tw, w \rangle = \langle T^*(T^*w), w \rangle = 0$, so $0 = Tw = T(T - T^*)v$.
As v is arbitrary, $T^2 = TT^*$.

Therefore, $T^* = T$.

4. Let V be a finite dimensional inner product space, and $T \in L(V)$ be invertible. Show that there exist $Q, P \in L(V)$ such that

   - Q is an isometry, that is $\| Qv \| = \| v \|$ for all $v \in V$
   - P is self-adjoint and positive definite, that is $\langle Pv, v \rangle > 0$ for all nonzero $v \in V$
   - $T = QP$

Solution: Since T^*T is self-adjoint, there exists an orthonormal eigenbasis $\alpha = \{ e_1, \ldots, e_n \}$ for T^*T.
Let the associated eigenvalues be $\mu_1, \ldots, \mu_n$.
Then, for each j, $\mu_j = \langle T^*Te_j, e_j \rangle = \langle Te_j, Te_j \rangle = \| Te_j \|^2 > 0$.

Let $P \in L(V)$ be the linear map on V that $Pe_j = \sqrt{\mu_j} e_j$ for each j.
By definition, $P^2 e_j = \mu_j e_j = T^*Te_j$ for each j, so $P^2 = T^*T$.
Since T is invertible, T^*T is invertible. As V is finite dimensional, P is also invertible.
Let $Q = TP^{-1}$.

We now verify that Q, P satisfy the requirements.
By definition, $QP = TP^{-1}P = T$.

Since α is an orthonormal basis, $P^*e_j = (\sqrt{\mu_j})^* e_j = \sqrt{\mu_j} e_j = Pe_j$ for each j, so $P^* = P$.
Let $v = \sum c_j e_j \in V$ be nonzero with $c_1, \ldots, c_n \in F$. Then $c_1, \ldots, c_n$ are not all zero.
Thus, $\langle Pv, v \rangle = \left\langle \sum c_j \sqrt{\mu_j} e_j, \sum c_k e_k \right\rangle = \sum \sqrt{\mu_j} \mid c_j \mid^2 > 0$.
As v is arbitrary, P is positive definite.

By definition, $QQ^* = TP^{-1}(TP^{-1})^* = T(P^*P)^{-1}T^* = T(T^*T)^{-1}T^* = \mathrm{Id}$.
As V is finite dimensional, Q is invertible with $Q^{-1} = Q^*$.
This implies that for each $v \in V$, $\| Qv \|^2 = \langle Qv, Qv \rangle = \langle Q^*Qv, v \rangle = \langle v, v \rangle = \| v \|^2$, so $\| Qv \| = \| v \|$.
Hence, Q is an isometry.

Note

This is the *polar decomposition* of a linear map and can be seen as a high dimension analog of the following property of complex number: for a nonzero complex number $t \in \mathbb{C} \setminus \{0\}$, $t = qp$ where $q = tp^{-1} \in \mathbb{C}$ has the property that $\mid qz \mid = \mid z \mid$ for all $z \in \mathbb{C}$, and $p = \sqrt{z^*z} \in \mathbb{C}$ is a real number which is also positive.

By appropriately defining Q, it is possible to extend this proposition to non-invertible maps, with P being positive semidefinite $\langle Pv, v \rangle \geq 0$. See Axler, Prop. 7.45, or textbook Theorem 6.28 for a version on matrices.

Note that $Q\alpha = \{ Qe_1, \ldots, Qe_n \}$ is still an orthonormal basis. This implies that there exists *two* orthonormal bases α, α' of V such that $[T]_\alpha^{\alpha'}$ is a diagonal matrix with nonnegative diagonal entries. This is the *singular value decomposition* of the operator T. See textbook Theorem 6.26, or Axler Prop. 7.51.