Topic#8 Invertibility & Isomorphism

Goal:

Let
$$T \in \mathcal{L}(V, W)$$

with $n = \dim(V) < \infty$ and $m = \dim(W) < \infty$
and α o.b. for V, β o.b. for W ,

then, T is bijective if and only if m = n and $[T]^{\beta}_{\alpha}$ is non-singular. Note:

 $A \in M_{n \times n}(F) \text{ is non-singular}$ $\iff det(A) \neq 0$ $\iff A \text{ is invertible}$ **<u>Def.</u>** $T \in \mathcal{L}(V, W)$. *T* is **invertible** if there exists a <u>function</u> $U: W \to V$ such that $TU = I_W$ and $UT = I_V$. Remark(1) T is invertible **iff** T is bijective. Pf:

$$\Rightarrow (a) T \text{ is onto. Indeed, let } y \in W,$$

then, $T(U(y) = TU(y) = I_W(y) = y$
i.e. $\exists U(u) \in V \text{ s.t. } T(U(y)) = y.$
(b) $T : V \rightarrow W$ is one-to-one. Indeed, let $T(x) = T(y), x, y \in V,$
then $U(T(x)) = U(T(y))$ i.e. $UT(x) = UT(y)$ i.e. $I_V(x) = I_V(y)$
i.e. $x = y$

 $\Leftarrow U \stackrel{def}{=} T^{-1}$

Remark(2) If T is invertible then U is unique, given $U = T^{-1}$.

Basic facts:

(1) If $T: V \to W$ is invertible then $T^{-1}: W \to V$ is invertible and $(T^{-1})^{-1} = T$.

(2) If $T : V \to W$ and $U : W \to Z$ are invertible, then $UT : V \to Z$ is invertible and $(UT)^{-1} = T^{-1}U^{-1}$. e.g.: Let $A \in M_{n \times n}(\mathbb{F})$, and $x \in \mathbb{F}^n \mapsto L_A(x) = Ax \in \mathbb{F}^n$ Then,

 $L_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ is invertible **iff** $A \in M_{n \times n}(\mathbb{F})$ is invertible.

In this case,

$$(L_A)^{-1} = L_{A^{-1}}.$$

<u>Thm.</u> If $T \in \mathcal{L}(V, W)$ is invertible

then $T^{-1}: W \to V$ is linear, so $T^{-1} \in \mathcal{L}(W, V)$.

Proof. Let $y_1, y_2 \in W$, $a \in \mathbb{F}$. $\therefore T$ is invertible $\therefore T$ is bijective, T^{-1} exists uniquely. $\therefore \exists ! x_1 = T^{-1}(y_1), x_2 = T^{-1}(y_2) \in V$, s.t. $y_1 = T(x_1), y_2 = T(x_2)$.

Then,

$$T^{-1}(a_1y_1 + a_2y_2)$$

= $T^{-1}(a_1T(x_1) + a_2T(x_2))$
= $T^{-1}(T(a_1x_1 + a_2x_2))$
= $a_1x_1 + a_2x_2$
= $a_1T^{-1}(y_1) + a_2T^{-1}(y_2)$

Lemma. Let $T \in \mathcal{L}(V, W)$ be invertible. Then $\dim(V) < \infty$ iff $\dim(W) < \infty$. In this case, $\dim(V) = \dim(W) < \infty$.

Proof. "
$$\Rightarrow$$
" Let dim $(V) < \infty$. Let β be a finite basis for V
 $W \stackrel{T \text{ is onto}}{=} R(T) = \text{span}(T(\beta)) \quad \therefore \dim(W) \le n < \infty.$

"
$$\Leftarrow$$
" Let dim $(W) < \infty$,
apply $T^{-1} \in \mathcal{L}(W, V)$ to show dim $(V) < \infty$.

Let dim(V) = $n < \infty$, $\beta = \{v_1, \dots, v_n\}$ a basis for V. Then W=span({ $T(v_1), \dots, T(v_n)$ })

Claim:
$$T(\beta) = \{T(v_1), \cdots, T(v_n)\}$$
 is I. indep.
(:: T is one-to-one).

If so, $T(\beta)$ is a basis for W. $\sharp T(\beta) = n \therefore \dim(W) = n = \dim(V)$.

Proof of claim: Let $\sum_{i=1}^{n} a_i T(v_i) = 0_v$ for $a_1, \dots, a_n \in F$. To show: $a_1 = \dots = a_n = 0$. $\therefore 0 = \sum_{i=1}^{n} a_i T(v_i) = T(\sum_{i=1}^{n} a_i v_i) \quad (\because T \in \mathcal{L})$ And T is one-to-one.

 $\therefore \sum_{i=1}^{n} a_i v = 0$ $\therefore \beta = \{v_1, \cdots, v_n\} \text{ is l.indep.}$ $\therefore a_1 = \cdots = a_n = 0.$ <u>**Thm.**</u> Let $T \in \mathcal{L}(V, W)$, where V, W are finite-dimensional with ordered bases β, γ , respectively. Then

T is invertible **iff** $[T]^{\gamma}_{\beta}$ is invertible.

Moreover,

$$[T^{-1}]^{\beta}_{\gamma} = ([T]^{\gamma}_{\beta})^{-1}.$$

<u>Proof.</u> " \Rightarrow " Assume: T is invertible. First, dim $(V) = \dim(W)$ by lemma. Let $n = \dim(V) = \dim(W)$. By $T^{-1}T = 1_V$,

$$I_{n\times n} = [I_V]^{\beta}_{\beta} = [T^{-1}T]^{\beta}_{\beta} = [T^{-1}]^{\beta}_{\gamma}[T]^{\gamma}_{\beta}$$

Similarly, by $TT^{-1} = I_W$,

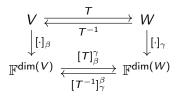
$$I_{n\times n} = [I_W]^{\gamma}_{\gamma} = [TT^{-1}]^{\gamma}_{\gamma} = [T]^{\gamma}_{\beta}[T^{-1}]^{\beta}_{\gamma}.$$

 \therefore $[\mathcal{T}]^{\gamma}_{\beta}$ is invertible, and $([\mathcal{T}]^{\gamma}_{\beta})^{-1} = [\mathcal{T}^{-1}]^{eta}_{\gamma}$.

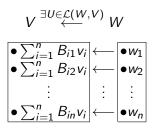
" \Leftarrow " Assume: $A \stackrel{def}{=} [T]^{\gamma}_{\beta}$ is invertible (of finite size) To show $T \in \mathcal{L}(V, W)$ is invertible. It suffices to show T is one to one.

Let $v_1, v_2 \in V$, and $T(v_1) = T(v_2)$. $\Rightarrow [T(v_1)]_{\gamma} = [T(v_2)]_{\gamma}$ $\Rightarrow [T]_{\beta}^{\gamma}[v_1]_{\beta} = [T]_{\beta}^{\gamma}[v_2]_{\beta}$ $\Rightarrow [v_1]_{\beta} = [v_2]_{\beta} (\because [T]_{\beta}^{\gamma} \text{ is invertible})$ $\Rightarrow v_1 = v_2.$

Remark:



T is invertible $\Leftrightarrow [T]^{\gamma}_{\beta}$ is invertible



such that $U(w_j) = \sum_{i=1}^n B_{ij}v_i, j = 1, \cdots, n$.

By def.: $[B_{ij}]_{n \times n} = [U]_{\gamma}^{\beta}$.

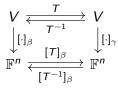
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Corollary. $T \in \mathcal{L}(V)$, where dim $(V) < \infty$ and β is an ordered basis for V. Then,

T is invertible **iff** $[T]_{\beta}$ is invertible.

Moreover, in this case,

$$[T^{-1}]_{\beta} = ([T]_{\beta})^{-1}.$$



<u>Def.</u> Let V, W: v.s. Then, V is **isomorphic** to W if there is an invertible $T \in \mathcal{L}(V, W)$.

In this case, T is called an **isomorphism** from V onto W.

<u>**Thm.**</u> Let V, W be finite-dimensional v.s.. Then,

V is isomorphic to *W* **iff** dim(*V*) = dim(*W*).

<u>Proof.</u> " \Rightarrow " Assume: *V* is isomorphic to *W*. $\therefore \exists$ an isomorphism $T \in \mathcal{L}(V, W)$ $\therefore T$ is invertible \therefore By the previous lemma, dim $(V) = \dim(W)$.

" \Leftarrow " Assume: dim $(V) = \dim(W) \stackrel{def}{=} n < \infty$. Let

$$\beta = \{v_1, \cdots, v_n\} : \text{ basis for V} \\ \gamma = \{w_1, \cdots, w_n\} : \text{ basis for } W$$

Then, $\exists ! T \in \mathcal{L}(V, W)$ such that $T(v_i) = w_i, i = 1, \dots, n$. Then $R(T) = \operatorname{span}(T(\beta)) = \operatorname{span}(\gamma) = W$ \therefore T is onto, hence one-to-one $(\dim(V) = \dim(W) < \infty)$ \therefore T is bijective. So $T \in \mathcal{L}(V, W)$ is invertible. So T is an isomorphism.

 \therefore V is isomorphic to W.

Corollary. Let V be a v.s. over \mathbb{F} . Then

V is isomorphic to \mathbb{F}^n iff dim(V) = n.

e.g. set dim(V)=n and β is an o.b. for V.

Write the standard representation of V under β as $[\cdot]_{\beta} = \phi_{\beta}$.

Then take any $v \in V$, see $[v]_{\beta} \in \mathbb{F}^n$ where $[v]_{\beta}$ is β -coordinate of $v \in V$.

The $[\cdot]_{\beta} : V \to \mathbb{F}^n$ is isomorphism.

<u>Def.</u> Let V be a v.s. over \mathbb{F} with dim(V) = n, and β be an ordered basis for V. The map

$$egin{aligned} \Phi_eta: \ V o \mathbb{F}^n \ v \mapsto \Phi_eta(v) \stackrel{def.}{=} [v]_eta \end{aligned}$$

is called the **standard representation** of V w.r.t. β .

<u>Note</u>: Φ_{β} is an isomorphism from V to \mathbb{F}^{n} .

<u>Thm.</u> Let V, W be finite-dimensional v.s. over \mathbb{F} with $\dim(V) = n$, $\dim(W) = m$, and ordered bases β , γ , resp.

$$V \xrightarrow{T \in \mathcal{L}(V,W)} W$$
$$\downarrow [:]_{\beta} \qquad \qquad \downarrow [:]_{\beta \in M_{m \times n}(\mathbb{F})} \qquad \qquad \downarrow [:]_{\beta \in M_{m \times n}(\mathbb{F})}$$
$$\mathbb{F}^{n} \xrightarrow{[T]_{\beta}^{\gamma} \in M_{m \times n}(\mathbb{F})} \mathbb{F}^{m}$$

Then, the mapping

$$\Phi: \mathcal{L}(V, W) \to M_{m \times n}(\mathbb{F})$$
$$T \mapsto \Phi(T) = [T]_{\beta}^{\gamma}$$

is an isomorphism (i.e. an invertible linear transformation). ($\therefore \mathcal{L}(V, W)$ is isomorphic to $M_{m \times n}(\mathbb{F})$).

This tells: $\mathcal{L}(V, W)$ is finite-dimensional with

 $\dim(\mathcal{L}(V,W)) = \dim(M_{m \times n}(\mathbb{F})) = mn.$

Proof.: 1°. Φ is well-defined and linear. Proved before. 2° Φ is one-to-one.

 $\Phi(T_1) = \Phi(T_2) \text{ i.e. } [T_1]_{\beta}^{\gamma} = [T_2]_{\beta}^{\gamma} \text{ to show: } T_1 = T_2$ take $v \in V$, $[T_1(v)]_{\gamma} = [T_1]_{\beta}^{\gamma}[v]_{\beta}$, $[T_2(v)]_{\gamma} = [T_2]_{\beta}^{\gamma}[v]_{\beta}$ $\therefore [T_1(v)]_{\gamma} = [T_2(v)]_{\gamma} \therefore T_1(v) = T_2(v).$

3°.
$$\Phi$$
 is onto. let $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{F})$.
To show: $\exists T \in \mathcal{L}(V, W)$ s.t. $A = \Phi(T) = [T]_{\beta}^{\gamma}$.
Indeed, $\beta = \{v_1, \cdots, v_n\}$ o.b. for V and $\gamma = \{w_1, \cdots, w_m\}$ o.b. for W .

Then, $\exists ! T \in \mathcal{L}(V, W)$ such that $T(v_j) = \sum_{i=1}^m a_{ij}w_i, 1 \le j \le n$. $\therefore A = [T]_{\beta}^{\gamma} = \Phi(T)$, i.e. T is onto.