SUGGESTED SOLUTIONS TO HOMEWORK 9

1. COMPULSORY PART

Exercise 1. In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

Solution. It suffices to prove that for $x \neq 0$, $\langle x, x\rangle > 0$. Indeed, denote $x = (x_1\ x_2)$, then

$$\begin{aligned}\langle x, x\rangle =&(x_1\ x_2)\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} \\ =&(x_1\ x_2)\begin{pmatrix} \overline{x_1} + i\overline{x_2} \\ -i\overline{x_1} + 2 \end{pmatrix} \\ =&|x_1|^2 + 2|x_2|^2 - 2\Im(x_1\overline{x_2}),\end{aligned}$$

where $\Im(x)$ denotes the imaginary part of x. Since

$$2\Im(x_1\overline{x_2}) \leq |x_1|^2 + |x_2|^2,$$

therefore

$$\langle x, x\rangle > 0.$$

By the definition,

$$\begin{aligned}\langle x, y\rangle =&(1-i\ 2+3i)\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 2-i \\ 3+2i \end{pmatrix} \\ =&6+21i.\end{aligned}$$

Exercise 2. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a, b), (c, d)\rangle = ac - bd$ on $\mathbb{R}^2$.

(b) $\langle A, B\rangle = tr(A + B)$ on $\mathsf{M}_{2\times 2}(\mathbb{R})$.

(c) $\langle f(x), g(x)\rangle = \int_0^1 f'(t)g(t)dt$ on $\mathsf{P}(\mathbb{R})$, where ' denotes differentiation.

Solution. (a) Since $\langle (1, 1), (1, 1)\rangle = 0$, but $(1, 1) \neq 0$.

(b) Since $\langle 2I_2, I_2\rangle = 3$, but $2\langle I_1, I_2\rangle = 4$.

(c) Since $\langle 1, 1\rangle = 0$, but $1 \neq 0$.

Exercise 3. Let V be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and let W be an inner product space over $\mathbb{F}$ with inner product $\langle \cdot, \cdot\rangle$. If $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ is linear, prove that $\langle x, y\rangle' = \langle \mathsf{T}(x), \mathsf{T}(y)\rangle$ defines an inner product on V if and only if T is one-to-one.

Solution. $\Leftarrow$: It is clear that $\langle \cdot, \cdot\rangle'$ is an inner product on V.

$\Rightarrow$: Suppose $\mathsf{T}(x_1) = \mathsf{T}(x_2)$, then

$$\langle x_1 - x_2, x_1 - x_2\rangle' = \langle \mathsf{T}(x_1 - x_2), \mathsf{T}(x_1 - x_2)\rangle = 0,$$

which implies that $x_1 = x_2$. Therefore T is one-to-one.

Exercise 4. Let V be a real or complex vector space (possibly infinite-dimensional), and let β be a basis for V. For $x, y \in \mathsf{V}$ there exists $v_1, v_2, ..., v_n \in \beta$ such that

$$x = \sum_{i=1}^{n} a_i v_i \quad \text{and} \quad y = \sum_{i=1}^{n} b_i v_i.$$

Define

$$\langle x, y \rangle = \sum_{i=1}^{n} a_i \overline{b_i}.$$

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V and that β is an orthonormal basis for V. Thus every real or complex vector space may be regarded as an inner product space.

(b) Prove that if $\mathsf{V} = \mathbb{R}^n$ or $\mathsf{V} = \mathbb{C}^n$ and β is the standard ordered basis, then the inner product defined above is the standard inner product.

Solution. (a) It is clear that $\langle \cdot, \cdot \rangle$ is an inner product. To prove β is an orthonormal basis, it suffices to note that

$$\langle v_i, v_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

(b) Let $\mathsf{V} = \mathbb{R}^n$ or $\mathbb{C}^n$ and β be the standard ordered basis, then for arbitrary $x = (x_1, ..., x_n) \in \mathsf{V}$ and $y = (y_1, ..., y_n) \in \mathsf{V}$, on the one hand, we have

$$x \cdot y = \sum_{i=1}^{n} x_i \bar{y}_i.$$

On the other hand, we have

$$x = \sum_{i=1}^{n} x_i e_i \quad \text{and} \quad y = \sum_{i=1}^{n} y_i e_i,$$

where $e_i = (0, ..., \underset{i\text{-th}}{1}, ..., 0)$, then

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Therefore $x \cdot y = \langle x, y \rangle$.

2. OPTIONAL PART

Exercise 5. Label the following statements as true or false.

(a) An inner product is a scalar-valued function on the set of ordered pairs of vectors.

(b) An inner product space must be over the field of real or complex numbers.

(c) An inner product is linear in both components.

(d) There is exactly one inner product on the vector space $\mathbb{R}^n$.

(e) The triangle inequality only holds in finite-dimensional inner product spaces.

(f) Only square matrices have a conjugate-transpose.

(g) If x, y and z are vectors in an inner product space such that $\langle x, y \rangle = \langle x, z \rangle$, then $y = z$.

(h) If $\langle x, y \rangle = 0$ for all x in an inner product space, then $y = 0$.

Solution. (a) True.
(b) True.
(c) False.
(d) False.
(e) False.
(f) False.
(h) True.

Exercise 6. Let $\{v_1, v_2, ..., v_k\}$ be an orthogonal set in V, and let $a_1, a_2, ..., a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

Solution. Since

$$\langle v_i, v_j \rangle = \begin{cases} \|v_i\|^2, & i = j, \\ 0, & i \neq j, \end{cases}$$

then

$$\begin{aligned} \left\| \sum_{i=1}^{k} a_i v_i \right\|^2 &= \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j \right\rangle \\ &= \sum_{i=1}^{k} a_i \left\langle v_i, \sum_{j=1}^{k} a_j v_j \right\rangle \\ &= \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2. \end{aligned}$$

Exercise 7. Let V be an inner product space. Prove that
(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re\langle x, y\rangle + \|y\|^2$ for all $x, y \in \mathsf{V}$, where $\Re\langle x, y\rangle$ denotes the real part of the complex number $\langle x, y\rangle$.
(b) $|\|x\| - \|y\|| \leq \|x - y\|$ for all $x, y \in \mathsf{V}$.

Solution. (a) Since

$$\begin{aligned} \|x + y\|^2 =& \langle x + y, x + y\rangle \\ =& \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ =& \|x\|^2 + 2\Re\langle x, y\rangle + \|y\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \|x - y\|^2 =& \|x\|^2 + 2\Re\langle x, -y\rangle + \|-y\|^2 \\ =& \|x\|^2 - 2\Re\langle x, y\rangle + \|y\|^2. \end{aligned}$$

(b) By the triangle inequality, on the one hand,

$$\|x\| \leq \|x - y\| + \|y\|,$$

which implies that

$$\|x\| - \|y\| \leq \|x - y\|.$$

On the other hand,

$$\|y\| \leq \|y - x\| + \|x\|,$$

which implies that

$$\|y\| - \|x\| \le \|x - y\|.$$

Therefore

$$|\|x\| - \|y\|| \le \|x - y\|.$$

Exercise 8. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*).$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?

(b) Let A be an $n \times n$ matrix. Prove that the representation in (a) is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

Solution. (a) Since

$$A_1^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + A) = A_1,$$
$$A_2^* = -\frac{1}{2}(iA - iA^*)^* = -\frac{1}{2}(-iA^* + iA) = A_2.$$

In addition,

$$A_1 + iA_2 = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = A.$$

(b) Suppose there exist B_1, B_2 with $B_1 = B_1^*$ and $B_2 = B_2^*$ such that

$$A = B_1 + iB_2.$$

Then

$$A^* = B_1^* - iB_2^* = B_1 - iB_2,$$

therefore

$$B_1 = \frac{1}{2}(A + A^*),$$
$$B_2 = \frac{1}{2i}(A - A^*).$$

Let $A = (a)$ where $a \in \mathbb{C}$, then

$$A_1 = \left(\frac{1}{2}(a + \bar{a})\right) = (\Re(a)),$$
$$A_2 = \left(\frac{1}{2i}(a - \bar{a})\right) = (\Im(a)).$$

In addition, A_1 and A_2 are uniquely defined, which implies it is reasonable to define A_1 and A_2 to be the real and imaginary part of A.

Exercise 9. Let $\mathsf{V} = \mathbb{F}^n$, and let $A \in \mathsf{M}_{n\times n}(\mathbb{F})$.

(a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in \mathsf{V}$.

(b) Suppose that for some $B \in \mathsf{M}_{n\times n}(\mathbb{F})$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathsf{V}$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathsf{T}(x) = Ax$ and $\mathsf{U}(x) = A^*x$. Show that $[\mathsf{U}]_\beta = [\mathsf{T}]_\beta^*$ for any orthonormal basis β for V.

Solution. (a) For all $x, y \in \mathsf{V}$, we have

$$\langle x, Ay \rangle = x^* Ay = (A^*x)^* y = \langle A^*x, y \rangle.$$

(b) Let $x = e_i$, $y = e_j$, where $e_i = (0, ..., \underset{i\text{th}}{1}, ..., 0)$, then

$$\langle e_i, Ae_j \rangle = a_{ij},$$
$$\langle Be_i, e_j \rangle = \overline{b_{ji}},$$

which implies $b_{ij} = \overline{a_{ji}}$, therefore $B = A^*$.

(c) We claim that $\langle Qx, Qy \rangle = \langle x, y \rangle$. Indeed,

$$\begin{aligned}\langle Qx, Qy \rangle =& \left\langle \sum_{i=1}^n x_i Q_i, \sum_{j=1}^n y_j Q_j \right\rangle \\ =& \sum_{i=1}^n x_i \left\langle Q_i, \sum_{j=1}^n y_j Q_j \right\rangle \\ =& \sum_{i=1}^n x_i \overline{y_i} \\ =& \langle x, y \rangle,\end{aligned}$$

therefore

$$\langle x, Qy \rangle = \langle QQ^{-1}x, Qy \rangle = \langle Q^{-1}x, y \rangle,$$

by (b), we have $Q^* = Q^{-1}$.

(d) Let Q be the $n \times n$ matrix whose columns are the vectors in β. Since

$$[\mathsf{U}]_\beta = Q^{-1} A^* Q,$$
$$[\mathsf{T}]_\beta = Q^{-1} A Q,$$

then

$$[\mathsf{T}]_\beta^* = (Q^{-1}AQ)^* = Q^* A^* (Q^{-1})^* = Q^{-1} A^* Q = [\mathsf{U}]_\beta.$$