SUGGESTED SOLUTIONS TO HOMEWORK 7

1. COMPULSORY PART

Exercise 1. Let $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$, $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$. Compute $[\mathsf{T}]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

Solution. Since

$$\mathsf{T}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\cdot\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$\mathsf{T}\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$\mathsf{T}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$\mathsf{T}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 0\cdot\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

therefore

$$[\mathsf{T}]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which implies that β is a basis consisting of eigenvectors of T.

Exercise 2. Let $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ and $\mathbb{F} = \mathbb{C}$.

(i) Determine all the eigenvalues of A.
(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
(iii) If possible, find a basis for $\mathbb{F}^2$ consisting of eigenvectors of A.
(iv) If successful in finding such a basis, determine an invertible matirx Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. (i) Since the characteristic polynomial of A is

$$\det(A - \lambda I_2) = (\lambda - 1)(\lambda + 1),$$

which implies that the eigenvalues of A are -1 and 1.

(ii) For the eigenvector corresponding to -1, consider

$$\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then we find the eigenspace E_{-1},

$$E_{-1} = \operatorname{span}\left\{ \begin{pmatrix} -1+i \\ 2 \end{pmatrix} \right\}.$$

For the eigenvector corresponding to 1, consider

$$\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then we find the eigenspace E_1,

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right\}.$$

(iii) By (ii), we have

$$\left\{ \begin{pmatrix} -1+i \\ 2 \end{pmatrix}, \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right\},$$

is a basis of $\mathbb{F}^2$ consisting of eigenvectors of A.

(iv) Let

$$Q = \begin{pmatrix} -1+i & 1+i \\ 2 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$Q^{-1}AQ = D.$$

Exercise 3. Let $\mathsf{V} = \mathsf{M}_{2\times 2}(\mathbb{R})$ and $\mathsf{T}(A) = A^t + 2 \cdot tr(A) \cdot I_2$. Find the eigenvalues of T and an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

Solution. Let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Since

$$\mathsf{T}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathsf{T}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathsf{T}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathsf{T}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

therefore

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}.$$

Since the characteristic polynomial of $[\mathsf{T}]_\alpha$ is

$$\det([\mathsf{T}]_\alpha - \lambda I_2) = (\lambda+1)(\lambda-1)^2(\lambda-5),$$

which implies that the eigenvalues of $[\mathsf{T}]_\alpha$ are -1, 1 and 5.

For the eigenvectors corresponding to 5, consider

$$\begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 5 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

then we find the eigenspace E_5,

$$E_5 = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}.$$

For the eigenvectors corresponding to 1, consider

$$\begin{pmatrix}3&0&0&2\\0&0&1&0\\0&1&0&0\\2&0&0&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

then we find the eigenspace E_1,

$$E_1 = \operatorname{span}\left\{\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\}.$$

For the eigenvectors corresponding to -1, consider

$$\begin{pmatrix}3&0&0&2\\0&0&1&0\\0&1&0&0\\2&0&0&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = -\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

then we find the eigenspace E_{-1},

$$E_{-1} = \operatorname{span}\left\{\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}\right\}.$$

Let

$$\begin{aligned}
M_1 &= 1\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix} + 1\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix},\\
M_2 &= -1\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix} + 1\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix},\\
M_3 &= 0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix} + 1\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} + 1\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix},\\
M_4 &= 0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix} + -1\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} + 1\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix} + 0\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix},
\end{aligned}$$

then we claim $[\mathsf{T}]_\beta$ for $\beta := \{M_1, M_2, M_3, M_4\}$. Indeed,

$$\mathsf{T}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} = 5\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 0\cdot\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$
$$\mathsf{T}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 1\cdot\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$
$$\mathsf{T}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 0\cdot\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + 1\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + 0\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$
$$\mathsf{T}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = 0\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 0\cdot\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} + 0\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + (-1)\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

therefore

$$[\mathsf{T}]_\beta = \begin{pmatrix}5 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

Exercise 4. Let V be a finite-dimensional vector space, and let λ be any scalar.
(a) For any ordered basis β for V, prove that $[\lambda \mathsf{I}_\mathsf{V}]_\beta = \lambda I$.
(b) Compute the characteristic polynomial of $\lambda \mathsf{I}_\mathsf{V}$.
(c) Show that $\lambda \mathsf{I}_\mathsf{V}$ is diagonalizable and has only one eigenvalue.

Solution. (a) Let $\beta = \{v^1, ..., v^n\}$ be an ordered basis of V, then

$$([\lambda \mathsf{I}_\mathsf{V}]_\beta)_{ij} = ([\lambda v^j]_\beta)_i = \lambda \delta_{ij},$$

where δ_{ij} is defined by

$$\delta_{ij} = \begin{cases}1, & i = j,\\ 0 & i \neq j.\end{cases}$$

Therefore $[\lambda \mathsf{I}_\mathsf{V}]_\beta = \lambda I$.

(b) The characteristic polynomial of $\lambda \mathsf{I}_\mathsf{V}$ is

$$\det(\lambda I - \mu I) = (\lambda - \mu)^n,$$

where $n = \dim \mathsf{V}$.

(c) By (b), $\lambda \mathsf{I}_\mathsf{V}$ has only one eigenvalue λ. Moreover, by (a), β is an ordered basis of V such that $[\lambda \mathsf{I}_\mathsf{V}]_\beta$ is a diagonal matrix.

Exercise 5. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

Solution. It suffices to prove that $a_0 = \det(A)$. Indeed, since

$$f(t) = \det(A - \lambda I_n),$$

then

$$a_0 = f(0) = \det(A).$$

Exercise 6. Let $\mathsf{V} = \mathsf{P}_3(\mathbb{R})$ and T is defined by $\mathsf{T}(f(x)) = f'(x) + f''(x)$, respectively. Test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

Solution. Let $\alpha = \{1, x, x^2, x^3\}$. Since

$$\begin{aligned}\mathsf{T}(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3,\\ \mathsf{T}(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3,\\ \mathsf{T}(x^2) &= 2x + 2 = 2 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3,\\ \mathsf{T}(x^3) &= 3x^2 + 6x = 0 \cdot 1 + 6 \cdot x + 3 \cdot x^2 + 0 \cdot x^3,\end{aligned}$$

therefore

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the characteristic polynomial of T is

$$\det([\mathsf{T}]_\alpha - \lambda I_4) = x^4,$$

which implies that the eigenvalues of T are 0 and 4.

For the eigenvectors corresponding to 0, consider

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0,$$

then we find the eigenspace E_0,

$$E_0 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Therefore T is not diagonalizable.

Exercise 7. Suppose that $A \in \mathsf{M}_{n\times n}(\mathbb{F})$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

Solution. Let E_{λ_1} and E_{λ_2} be the eigenspaces corresponding to λ_1 and λ_2 respectively. Then $E_{\lambda_1} \cup E_{\lambda_2}$ is a linearly independent subset of $\mathbb{F}^n$. On the one hand,

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \geq n - 1 + 1 = n,$$

on the other hand,

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \leq \dim(E_{\lambda_1} \cup E_{\lambda_2}) \leq \dim \mathbb{F}^n = n,$$

therefore $E_{\lambda_1} \cup E_{\lambda_2}$ is a basis of $\mathbb{F}^n$.

Exercise 8. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Solution. (a) Let E_λ be the eigenspace corresponding to λ of T, and $E_{\lambda^{-1}}$ be the eigenspace corresponding to λ^{-1} of T^{-1}.

For arbitrary $v \in E_\lambda$, then

$$\mathsf{T}(v) = \lambda v,$$

therefore

$$\mathsf{T}^{-1}(v) = \mathsf{T}^{-1}(\lambda^{-1}\mathsf{T}(v)) = \lambda^{-1}v,$$

which implies that $v \in E_{\lambda^{-1}}$.

Similarly, for arbitrary $v \in E_{\lambda^{-1}}$, we have $v \in E_\lambda$.

Therefore $E_\lambda = E_{\lambda^{-1}}$.

(b) Since T is diagonalizable, then there exists an ordered basis β of V consisting of eigenvectors of T. By (a), we have β also consists of eigenvectors of T^{-1}, therefore T^{-1} is also diagonalizable.

2. OPTIONAL PART

Exercise 9. Label the following statements as true or false.

(a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

(b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

(c) There exists a square matrix with no eigenvectors.

(d) Eigenvalues must be nonzero scalar.

(e) Any two eigenvectors are linearly independent.

(f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.

(h) An $n \times n$ matrix A with entries from a field $\mathbb{F}$ is similar to a diagonal matix if and only if there is a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

(i) Similar matrices always have the same eigenvalues.

(j) Similar matrices always have the same eigenvectors.

(k) The sum of two eigenvectors of an operator T is always an eigenvector of T.

Solution. (a) False.

(b) True.

(c) True.

(d) False.

(e) False.

(f) False.

(h) True.

(i) True.

(j) False.

(k) False.

Exercise 10. Let $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$, $\mathsf{T}(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2$, and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$. Compute $[\mathsf{T}]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

Solution. Since

$$\begin{aligned}\mathsf{T}(x - x^2) &= 4 + 4x - 4x^2 = 4 \cdot 1 + 4 \cdot x + (-4) \cdot x^2,\\ \mathsf{T}(-1 + x^2) &= 2 - 2x^2 = 2 \cdot 1 + 0 \cdot x + (-2) \cdot x^2,\\ \mathsf{T}(-1 - x + x^2) &= 3x - 3x^2 = 0 \cdot 1 + 3 \cdot x + (-3) \cdot x^2,\end{aligned}$$

therefore

$$[\mathsf{T}]_\beta = \begin{pmatrix} 4 & 2 & 0 \\ 4 & 0 & 3 \\ -4 & -2 & -3 \end{pmatrix}.$$

which implies that β is not a basis consisting of eigenvectors of T.

Exercise 11. Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ and $\mathbb{F} = \mathbb{R}$.

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for $\mathbb{F}^3$ consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matirx Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution. (i) Since the characteristic polynomial of A is

$$\det(A - \lambda I_3) = -x(x-1)^2,$$

which implies that the eigenvalues of A are 0 and 1.

(ii) For eigenvectors corresponding to 0, consider

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

then we find the eigenspace E_0,

$$E_0 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

For the eigenvectos corresponding to 1, consider

$$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

then we find the eigenspace E_1,

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(iii) By (ii), let

$$\epsilon := \left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Then ϵ is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

(iv) Let

$$Q := \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 12. A scalar matrix is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Solution. (a) Since A is similar to a scalar matrix λI, then there exists an invertible matrix Q such that

$$Q^{-1}AQ = \lambda I,$$

which implies

$$A = Q(\lambda I)Q^{-1} = \lambda I.$$

(b) On the one hand, let M be a diagonalizable matrix having only one eigenvalue, then M is similar to a scalar matrix, by (a), we have M is a scalar matrix.

On the other hand, if M is a scalar matrix, then M is diagonalizable and M has only one eigenvalue.

(c) Let $A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, since the characteristic polynomial of A is

$$\det(A - \lambda I_2) = (\lambda - 1)^2,$$

which implies that the eigenvalue of A is 1.

For the eigenvectors corresponding to 1, consider

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then we find the eigenspace E_1,

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\},$$

which implies that A is not diagonalizable.

Exercise 13. Let T be the linear operator on $\mathsf{M}_{n\times n}(\mathbb{R})$ defined by $\mathsf{T}(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(\mathbb{R})$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(\mathbb{R})$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

Solution. (a) Let $\lambda \in \mathbb{R}$ be an eigenvalue of T, then there exists an eigenvector $M \in \mathsf{M}_{n\times n}(\mathbb{R})$ such that

$$\mathsf{T}(M) = \lambda M,$$

which implies

$$M^t = \lambda M,$$

therefore

$$M_{ji} = \lambda M_{ij},$$

then

$$M_{ij} = \lambda M_{ji} = \lambda^2 M_{ij}.$$

Since $M \neq 0$, therefore

$$\lambda^2 = 1,$$

which implies $\lambda = \pm 1$.

(b) For the eigenvector M_1 corresponding to 1, M_1 is a symmetry matrix.

For the eigenvector M_{-1} corresponding to -1, M_1 is an skew-symmetry matrix.

(c) By (b), let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Then $[\mathsf{T}]_\beta$ is a diagonal matrix.

(d) By (b), let

$$\beta = \{E_{ii}\}_{i=1,\dots,n} \cup \{E_{ij} + E_{ji}\}_{i>j} \cup \{E_{ij} - E_{ji}\}_{i>j},$$

where E_{ij} is the matrix with its ij-entry 1 and all other entries 0.

Exercise 14. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that $f(t) = (A_{11} - t)(A_{22} - t) \cdots (A_{nn} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n - 2$.

(b) Show that $tr(A) = (-1)^{n-1} a_{n-1}$.

Solution. It suffices to prove (b). By Schur decomposition, there exists an invertible matrix $Q \in \mathsf{M}_{n\times n}$ such that

$$Q^{-1} A Q = U,$$

where $U \in \mathsf{M}_{n\times n}$ is an upper triangular matrix. Then

$$\det(A - tI_n) = \det(U - tI_n) = (U_{11} - t) \cdots (U_{nn} - t),$$

which implies

$$tr(A) = tr(U) = (-1)^{n-1} a_{n-1}.$$

Exercise 15. Label the following statements as true or false.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(c) If λ is an eigenvalue of a linear operator T, then each vector in $\mathsf{E}_\lambda$is an eigenvector of T.

(d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $\mathsf{E}_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

(e) Let $A \in \mathsf{M}_{n\times n}(\mathbb{F})$ and $\beta = \{v_1, v_2, ..., v_n\}$ be an ordered basis for $\mathbb{F}^n$ consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

The following two items relate to the optional subsection on direct sums

(h) If a vector space is the direct sum of subspaces $\mathsf{W}_1, \mathsf{W}_2, ..., \mathsf{W}_k$, then $\mathsf{W}_i \cap \mathsf{W}_j = \{0\}$ for $i \neq j$.

(i) If

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \quad \text{and} \quad \mathsf{W}_i \cap \mathsf{W}_j = \{0\} \quad \text{for } i \neq j,$$

then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

Solution. (a) False.
(b) False.
(c) False.
(d) True.
(e) True.
(f) False.
(g) True.
(h) True.
(i) False.

Exercise 16. Let $\mathsf{V} = \mathbb{C}^2$ and T is defined by $\mathsf{T}(z, w) = (z + iw, iz + w)$. Test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

Solution. (a) Let $\alpha = \{(1, 0), (0, 1)\}$, then

$$\mathsf{T}(1, 0) = (1, i) = 1 \cdot (1, 0) + i \cdot (0, 1),$$
$$\mathsf{T}(0, 1) = (i, 1) = i \cdot (1, 0) + 1 \cdot (0, 1),$$

therefore

$$[\mathsf{T}]_\alpha = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Since the characteristic polynomial of T is

$$\det([\mathsf{T}]_\alpha - \lambda I_2) = (\lambda - 1 - i)(\lambda - 1 + i),$$

which implies that the eigenvalues of T are $1 + i$ and $1 - i$.

For the eigenvectors corresponding $1 - i$, consider

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1 - i) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then we find the eigenspace E_{1-i},

$$E_{1-i} = \operatorname{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

For the eigenvectors corresponding $1 + i$, consider

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1 + i) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

then we find the eigenspace E_{1+i},

$$E_{1+i} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Let $\beta := \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 - i & 0 \\ 0 & 1 + i \end{pmatrix}.$$

Exercise 17. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ and corresponding multiplicities $m_1, m_2, ..., m_k$. Suppose that β is a basis for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[T]_\beta$ are $\lambda_1, \lambda_2, ..., \lambda_k$ and that each λ_i occurs m_i times $(1 \leq i \leq k)$.

Solution. Let $n = \dim \mathsf{V}$. Then the characteristic polynomial of T is,

$$\det([\mathsf{T}]_\beta - \lambda I_n) = (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_2 - \lambda)^{m_k},$$

which implies the diagonal entries of $[T]_\beta$ are $\lambda_1, \lambda_2, ..., \lambda_k$ and that each λ_i occurs m_i times $(1 \leq i \leq k)$.

Exercise 18. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ with corresponding multiplicities $m_1, m_2, ..., m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$.

(b) $\det(A) = (\lambda_1)^{m_1} (\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}$.

Solution. Since the characteristic polynomial of T is,

$$\det([\mathsf{T}]_\beta - \lambda I_n) = (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k},$$

therefore

$$\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i,$$

and

$$\det(A) = (\lambda_1)^{m_1} (\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}.$$

Exercise 19. (a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[T]_\beta$ and $[\mathsf{U}]_\beta$ are simultaneously diagonalizable for any ordered basis β.

(b) Prove that if A and B are simultaneously diagonalizable matrices in $\mathsf{M}_{n\times n}(\mathbb{F})$, then L_A and L_B are simultaneously diagonalizable linear operators.

Solution. (a) Let $n = \dim \mathsf{V}$, $\alpha := \{v^1, ..., v^n\}$ be an ordered basis of V such that T and U are simultaneously diagonalizable, then $[\mathsf{T}]_\alpha$ and $[\mathsf{U}]_\alpha$ are diagonal matrices. Let $\beta := \{w^1, ..., w^n\}$ be an arbitrary ordered basis of V, then there exists an invertible matrix$Q \in \mathsf{M}_{n\times n}(\mathbb{F})$ such that

$$[w^i]_\Lambda = \sum_{j=1}^{n} [v^j]_\Lambda Q_{ji},$$

for $\Lambda = \alpha, \beta$. Then

$$\begin{aligned}
[v^i]_\Lambda &= \sum_{k=1}^{n} [v^k]_\Lambda (I_n)_{ki} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} [v^k]_\Lambda Q_{kj} (Q^{-1})_{ji} \\
&= \sum_{j=1}^{n} [w^j]_\Lambda (Q^{-1})_{ji}.
\end{aligned}$$

Let λ_i be the eigenvalue corresponding to the eigenvector $v^i \in \alpha$, therefore

$$\begin{aligned}([\mathsf{T}]_\beta)_{ij} =&([\mathsf{T}(w^j)]_\beta)_i\\ =&\sum_{k=1}^n([\lambda_k v^k]_\beta)_i Q_{kj}\\ =&\sum_{k=1}^n\sum_{l=1}^n \lambda_k([w^l]_\beta)_i(Q^{-1})_{lk}Q_{kj}\\ =&(Q^{-1}\text{diag}(\lambda_1,...,\lambda_k)Q)_{ij},\end{aligned}$$

which implies

$$[\mathsf{T}]_\beta = Q^{-1}[\mathsf{T}]_\alpha Q.$$

Similarly, $[\mathsf{U}]_\beta = Q^{-1}[\mathsf{U}]_\alpha Q$.

(b) Since A and B are simultaneously diagonalizable matrices, then there exists an invertible matrix Q such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are simultaneously diagonal matrices. Let $\beta := \{Q^1, ..., Q^n\}$ where Q^i is the i-th column of Q, then we claim that $[\mathsf{L}_A]_\beta$ and $[\mathsf{L}_B]_\beta$ are diagonal matrices. Indeed,

$$([\mathsf{L}_A]_\beta)_{ij} = (AQ^j)_i = \lambda_j Q_{ij},$$

which implies that $[\mathsf{L}_A]_\beta$ is a diagonal matrix. Similarly, $[\mathsf{L}_B]_\beta$ is a diagonal matrix

Exercise 20. (a) Prove that if T and U are simultaneously diagonalizable operators on a finite-dimensional vector space V, then T and U commute.

(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B are commute.

Solution. (a) Let $n = \dim \mathsf{V}$ and $\beta := \{v^1, ..., v^n\}$ be an ordered basis such that $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are diagonal matrices. Then for arbitrary $v^i \in \beta$, we have

$$\mathsf{TU}(v^i) - \mathsf{UT}(v^i) = \mathsf{T}(b_i v^i) - \mathsf{U}(a_i v^i) = b_i a_i v^i - a_i b_i v^i = 0.$$

For arbitrary $v \in \mathsf{V}$, since there exists $\lambda_1, ..., \lambda_n$ such that

$$v = \lambda_1 v^1 + \cdots + \lambda_n v^n,$$

therefore

$$\mathsf{TU}(v) - \mathsf{UT}(v) = \lambda_1(\mathsf{TU}(v^1) - \mathsf{UT}(v^1)) + \cdots + \lambda_n(\mathsf{TU}(v^n) - \mathsf{UT}(v^n)) = 0,$$

which implies that T and U are commute.

(b) Since A and B are simultaneously diagonalizable matrices, then there exists an invertible matrix Q such that $Q^{-1}AQ = \Lambda_1$ and $Q^{-1}BQ = \Lambda_2$ where Λ_1 and Λ_2 are diagonal matrices. Therefore

$$AB = Q\Lambda_1\Lambda_2 Q^{-1} = Q\Lambda_2\Lambda_1 Q^{-1} = BA.$$

Exercise 21. Let T be a diagonalizable linear operators on a finite-dimensional vector space V, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

Solution. Let $\beta := \{v^1, ..., v^n\}$ be an ordered basis of V such $[\mathsf{T}]_\beta$ is a diagonal matrix. Since

$$\begin{aligned}([\mathsf{T}^m]_\beta)_{ij} =& ([\mathsf{T}^m(v^j)]_\beta)_i, \\ =& \lambda_j([\mathsf{T}^{m-1}(v^j)]_\beta)_i \\ =& \lambda_j^{m-1}([T(v^j)]_\beta)_i \\ =& \lambda_j^{m-1}([\mathsf{T}]_\beta)_{ij},\end{aligned}$$

where λ_j is the eigenvalue corresponding to v^j, therefore T and T^m are simultaneously diagonalizable.