SUGGESTED SOLUTIONS TO HOMEWORK 11

1. SECTION 6.4

Exercise 1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every self-adjoint operator is normal.

(b) Operators and their adjoints have the same eigenvectors.

(c) If T is an operator on an inner product space V, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal, where β is any ordered basis for V.

(d) A real or complex matrix A is normal if and only if L_A is normal.

(e) The eigenvalues of a self-adjoint operator must all be real.

(f) The identity and zero operators are self-adjoint.

(g) Every normal operator is diagonalizable.

(h) Every self-adjoint operator is diagonalizable.

Solution. (a) True.

(b) False.

(c) False.

(d) True.

(e) True.

(f) True.

(g) False.

(h) True.

Exercise 2. Let $\mathsf{T}(f) = f'$ on the linear space $\mathsf{V} = \mathsf{P}_2(\mathbb{R})$ with inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

Solution. Let $\beta := \{1, \sqrt{3}(2t-1), \sqrt{6}(6t^2-6t+1)\}$, then β is an orthonormal basis of V. Since

$$\begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 6\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix},$$

therefore T is neither normal nor self-adjoint.

Exercise 3. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant.

Solution. Since T is a normal operator and V is a finite-dimensional complex inner product space, then there exists an orthonormal basis β for V consisting of eigenvectors of T. Denote $n := \dim \mathsf{V}$ and $\beta := \{v_1, ..., v_n\}$, then there exist $\lambda_1, ..., \lambda_n \in \mathbb{C}$ such that $\mathsf{T}v_i = \lambda_i v_i$ for $i = 1, ..., n$. Since W is a subspace of V and it is T-invariant, then without lose of generality, we assume $\text{span}\{v_1, ..., v_m\} = \mathsf{W}$ for $m \in \mathbb{N}$ and $1 \le m \le n$. Let $w \in \mathsf{W}$, then there exist $\alpha_1, ..., \alpha_m \in \mathbb{C}$ such that

$$w = \sum_{i=1}^{m} \alpha_i v_i.$$

Therefore by $\mathsf{T}^* v_i = \bar{\lambda}_i v_i$ for $i = 1, ..., n$, we have

$$\mathsf{T}^*(w) = \sum_{i=1}^{m} \alpha_i \bar{\lambda}_i v_i \in \mathsf{W},$$

which implies that W is also T^*-invariant.

Exercise 4. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $\mathsf{N}(\mathsf{T}) = \mathsf{N}(\mathsf{T}^*)$ and $\mathsf{R}(\mathsf{T}) = \mathsf{R}(\mathsf{T}^*)$.

Solution. Let $v \in \mathsf{N}(\mathsf{T})$, then $\mathsf{T}(v) = 0$, which implies that v is an eigenvector of T corresponding to eigenvalue 0, therefore v is also an eigenvector of T^* corresponding to 0. This proves that $\mathsf{N}(\mathsf{T}) \subset \mathsf{N}(\mathsf{T}^*)$. Similarly, we also have $\mathsf{N}(\mathsf{T}^*) \subset \mathsf{N}(\mathsf{T})$ which implies that $\mathsf{N}(\mathsf{T}) = \mathsf{N}(\mathsf{T}^*)$.

Let $w \in \mathsf{R}(\mathsf{T})$, then there exists $v_w \in \mathsf{V}$ such that $w = \mathsf{T}(v_w)$, therefore for $v \in \mathsf{N}(\mathsf{T})$, we have

$$\langle w, v \rangle = \langle v_w, \mathsf{T}^*(v) \rangle,$$

since $\mathsf{N}(\mathsf{T}) = \mathsf{N}(\mathsf{T}^*)$, then

$$\langle w, v \rangle = 0,$$

which implies that $w \in \mathsf{N}(\mathsf{T})^\perp$. Since $\mathsf{R}(\mathsf{T}^*) = \mathsf{N}(\mathsf{T})^\perp$, then we have $\mathsf{R}(\mathsf{T}) \subset \mathsf{R}(\mathsf{T}^*)$. Similarly, we also have $\mathsf{R}(\mathsf{T}^*) \subset \mathsf{R}(\mathsf{T})$ which implies that $\mathsf{R}(\mathsf{T}) = \mathsf{R}(\mathsf{T}^*)$.

Exercise 5. Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence prove that T is self-adjoint.

Solution. It suffices to prove that V has an orthonormal basis of eigenvectors of T. Denote $n := \dim \mathsf{V}$. Since the characteristic polynomial of V splits, then by Schur's theorem, there exists an orthonormal basis $\beta = \{v_1, ..., v_n\}$ for V such that $[\mathsf{T}]_\beta = U$ is upper triangular. Let us use induction on $n \in \mathbb{N}$ to prove β is also the set of eigenvectors of T.

For $n = 1$, we clearly have that v_1 is an eigenvector of T.

Suppose there exists $k \in \mathbb{N}$ such that for all real inner product space V with $1 \le n \le k-1$, β is the set of eigenvectors of T . Consider $n = k$, since $\text{span}\{v_1, ..., v_{k-1}\}$ is an inner product space with dimension $k-1$, by the induction hypothesis, we have $v_1, ..., v_{k-1}$ are eigenvectors of T. By Schur's theorem, we have

$$\mathsf{T}(v_j) = U_{jj} v_j,$$

for $1 \le j \le k-1$. Then

$$\mathsf{T}^*(v_j) = \bar{U_{jj}} v_j,$$

for $1 \le j \le k-1$. Moreover, we have

$$\mathsf{T}(v_k) = \sum_{j=1}^{k} U_{jk} v_j,$$

then

$$U_{jk} = \langle \mathsf{T}(v_k), v_j \rangle = \langle v_k, \bar{U_{jj}} v_j \rangle = 0,$$

for $1 \leq j \leq k-1$, therefore

$$\mathsf{T}(v_k) = U_{kk} v_k,$$

which implies that v_k is also an eigenvectors of T. This proves that β is the set of eigenvectors of T.

Exercise 6. Let V be a finite-dimensional real inner product space, and let U and T be self-adjoint linear operators on V such that $\mathsf{UT} = \mathsf{TU}$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both U and T.

Solution. Let us use induction on the dimension $n \in \mathbb{N}$ of V.

For $n = 1$, the conclusion is clearly true.

Suppose the conclusion is true for $1 \leq n \leq k-1$ for some $k \in \mathbb{N}$. Consider $n = k$. Let E_λ be an eigenspace of T with eigenvalue λ. Then E_λ is T-invariant and U-invariant. Indeed, it is clear that E_λ is T-invariant, in addition, for $v \in \mathsf{E}_\lambda$, we have

$$\mathsf{TU}(v) = \mathsf{UT}(v) = \lambda \mathsf{U}(v),$$

which implies that $\mathsf{U}(v) \in \mathsf{E}_\lambda$. If $\mathsf{E}_\lambda = \mathsf{V}$, then since U are self-adjoint operator, there exists an orthonormal basis β for V consisting eigenvectors of U. Moreover, by the choice of V, it is clear that β is also a set of eigenvectos of T. This proves β is an orthonormal basis for Vconsisting of eigenvectors of U and T. Otherwise, for $\mathsf{E}_\lambda \subsetneq \mathsf{V}$, then by the induction hypothesis, there exists an orthonormal basis β' for $\mathsf{E}_\lambda$consisting of eigenvectors of $\mathsf{UI}_{\mathsf{E}_\lambda}$ and $\mathsf{TI}_{\mathsf{E}_\lambda}$, where $\mathsf{I}_{\mathsf{E}_\lambda}$ is the projection from V to E_λ. Moreover, let us consider $(\mathsf{E}_\lambda)^\perp$, we claim that $(\mathsf{E}_\lambda)^\perp$ is also T-invariant and U-invariant. Indeed, it is clear that $(\mathsf{E}_\lambda)^\perp$ is T-invariant, in addition, for $w \in (\mathsf{E}_\lambda)^\perp$ and $v \in \mathsf{E}_\lambda$, since E_λ is U-invariant, we have

$$\langle \mathsf{U}(w), v \rangle = \langle w, \mathsf{U}(v) \rangle = 0,$$

which implies that $\mathsf{U}(v) \in (\mathsf{E}_\lambda)^\perp$. Then by the induction hypothesis, there exists an orthonormal basis β'' for $(\mathsf{E}_\lambda)^\perp$ consisting of eigenvectors of $\mathsf{U}(\mathsf{I} - \mathsf{I}_{\mathsf{E}_\lambda})$ and $\mathsf{T}(\mathsf{I} - \mathsf{I}_{\mathsf{E}_\lambda})$. Therefore let $\beta := \beta' \cup \beta''$, we have β is an orthonormal basis for V consisting of eigenvectors of U and T.

2. SECTION 6.5

Q1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every unitary operator is normal.
(b) Every orthogonal operator is diagonalizable.
(c) A matrix is unitary if and only if it is invertible.
(d) If two matrices are unitarily equivalent, then they are also similar.
(e) The sum of unitary matrices is unitary.
(f) The adjoint of a unitary operator is unitary.
(g) If T is an orthogonal operator on V, then $[T]_\beta$ is an orthogonal matrix for any ordered basis β for V.
(h) If all the eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.
(i) A linear operator may preserve the norm, but not the inner product.

Sol: (a) True.
(b) False.

(c) False.
(d) True.
(e) False.
(f) Ture.
(g) False.
(h) False.
(i) False.

Q2(c). For the following matrix A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

$$\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$$

Sol. The characteristic polynomial of A is

$$(2-t)(5-t) - (3-3i)(3+3i) = t^2 - 7t - 8 = (t-8)(t+1).$$

Hence, $-1, 8$ are all the eigenvalues of A. Note that for any scalars a, b,

$$3\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (A-8I)\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

if and only if $b = (1+i)a$. In particular, $u = (1, 1+i)$ is an eigenvector of A corresponding to eigenvalue 8.

$$\|u\| = \sqrt{1\bar{1} + (1+i)\overline{(1+i)}} = \sqrt{3}.$$

On the other hand, for any scalars a, b,

$$3\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 3-3i \\ 3+3i & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (A+I)\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

if and only if $a = (i-1)b$. In particular, $v = (i-1, 1)$ is an eigenvector of A corresponding to eigenvalue -1.

$$\|v\| = \sqrt{(i-1)\overline{(i-1)} + 1\bar{1}} = \sqrt{3}.$$

Then

$$P = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & i-1 \\ i+1 & 1 \end{pmatrix}$$

is a unitary matrix and

$$D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$$

is a diagonal matrix such that $P^*AP = D$.

Q7. Prove if T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary square root.

Sol. Let β be the standard ordered basis and $A = [T]_\beta$. By Theorem 6.19 we have a unitary matrix Q and a diagonal matrix D s.t.

$$A = Q^*DQ.$$

Since A is unitary, we have $A^*A = Q^*D^*QQ^*DQ = Q^*D^*DQ = I$ which implies $D^*D = I$. By the fact that D is diagonal, denote

$$D = \begin{pmatrix} |d_1|e^{i\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |d_n|e^{i\theta_n}. \end{pmatrix}$$

Then we have $|d_j| = 1$. Let $U = Q^* \begin{pmatrix} \sqrt{|d_1|}e^{\frac{i\theta_1}{2}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{|d_n|}e^{\frac{i\theta_n}{2}} \end{pmatrix} Q$. We can varify U satiesfies our requirements.

Q10. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i \qquad \mathrm{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2,$$

where the λ_i's are the eigenvalues of A.

Sol. There are unitary matrix Q and diagonal matrix D s.t. $A = Q^*DQ$ and the diagonal elements of D are eigenvalues of A. Then we have

$$\mathrm{tr}(A) = \mathrm{tr}(Q^*DQ) = \mathrm{tr}(Q^*QD) = \mathrm{tr}(D) = \sum_{i=1}^{n} \lambda_i.$$

$$\mathrm{tr}(A^*A) = \mathrm{tr}(Q^*D^*QQ^*DQ) = \mathrm{tr}(D^*D) = \mathrm{tr}(H) = \sum_{i=1}^{n} |\lambda_i|^2.$$

Q12. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\det(A) = \prod_{i=1}^{n} \lambda_i,$$

where the λ_i's are the (not necessarily distinct) eigenvalues of A.

Sol. There are unitary matrix Q and diagonal matrix D s.t. $A = Q^*DQ$ and the diagonal elements of D are eigenvalues $\{\lambda_i\}_{i=1}^n$ of A. Then

$$\det(A) = \det(Q^*DQ) = \det(D) = \prod_{i=1}^{n} \lambda_i.$$

Q15. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that
(a) $U(W) = W$; [(b)] $W^\perp$ is U-invariant.

Sol. (a) Since U is W-invariant, we have $U(W) \subseteq W$. It then suffices to show that $W \subseteq U(W)$.
Consider $U_W : W \to W$, the restriction of U on W. Then U_W is linear. As U is unitary, $\|U(v)\| = \|v\|$ for all $v \in V$. In particular, $\|U_W(w)\| = \|U(w)\| = \|w\|$ for all $w \in$. So U_W is one-to-one. As W is finite dimensional, U_W is then onto, and so $W \subseteq U_W(W) = U(W)$.
Hence we have $U(W) = W$.

(b) Let $v \in W^\perp$. Then $\langle v, w\rangle = 0$ for all $w \in W$. Let $w \in W$. By the previous question there exists $w' \in W = U(W)$ such that $w = Uw'$. Then $\langle Uv, w\rangle = \langle v, U^*Uw'\rangle = \langle v, w'\rangle = 0$. As $w' \in W$ is arbitrary, $Uv \in W^\perp$.
As $v \in W^\perp$ is arbitrary, $W^\perp$ is U-invariant.