MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to Problem Set 3

Problems to hand in

1. Find the volume of the region lying above the plane $z = a$ and inside the sphere $x^2+y^2+z^2 = 4a^2$ by integrating in cylindrical coordinates and spherical coordinates.

Solution. Using cylindrical coordinates,

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^{\sqrt{3}a} \int_a^{\sqrt{4a^2-r^2}} r \, dz dr d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{3}a} \left(r\sqrt{4a^2 - r^2} - ar \right) \, dr d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{2} \cdot \frac{2}{3}(4a^2 - r^2)^{\frac{3}{2}} - \frac{a}{2}r^2 \right]_0^{\sqrt{3}a} \, d\theta \\
&= 2\pi \left[\frac{(4a^2)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}}{3} - \frac{3a^3}{2} \right] \\
&= \frac{5\pi a^3}{3}.
\end{aligned}$$

Using spherical coordinates,

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{a\sec\phi}^{2a} \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin\phi \cdot \frac{1}{3} \left(8a^3 - a^3 \sec^3 \phi \right) \, d\phi d\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} \left[-8\cos\phi - \frac{1}{2\cos^2\phi} \right]_0^{\frac{\pi}{3}} \, d\theta \\
&= \frac{5\pi a^3}{3}.
\end{aligned}$$

□

2. (a)Find the volume of a right circular cone of base radius a and height h by integrating in cylindrical coordinates and spherical coordinates.

   (b) How about the volume of an oblique cone where the vertex also lies at height h but not necessarily directly over the center of the circular base?

   (c) In general, what is the volume of a generalized cone with a given base area A and height h?

Solution. (a)Using cylindrical coordinates,

$$\text{Volume} = \int_0^{2\pi} \int_0^a \int_0^{h-\frac{rh}{a}} r \, dz dr d\theta = \cdots = \frac{1}{3}\pi a^2 h.$$

Using spherical coordinates,

$$\text{Volume} = \int_0^{2\pi} \int_0^{\arctan(\frac{a}{h})} \int_0^{h\sec\phi} \rho^2 \sin\phi \, d\rho d\phi d\theta = \cdots = \frac{1}{3}\pi a^2 h.$$

(b) and (c) can be done together . let $S(t)$ be area of section with heigth t , easily $S(0) = A, S(h) = 0$.

$$Vol(cone) = \int_0^h S(t)dt = \int_0^h S(h-t)dt \tag{1}$$

on the other hand $S(h-t)/S(o) = (\frac{t}{h})^2$. so we have

$$Vol(cone) = \int_0^h S(t)dt = \int_0^h S(h-t)dt = \int_0^h (\frac{t}{h})^2 A dt = Ah/3 \tag{2}$$

so for (b) , volume equal to $\pi a^2 h/3$ □

3. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2 + y^2 = 1$, $y^2 + z^2 = 1$, and $x^2 + z^2 = 1$.

Solution. By symmetry,

$$\begin{aligned}\text{Volume} &= 8 \int_0^{\frac{\pi}{4}} \int_0^1 \int_{-\sqrt{1-r^2\cos^2\theta}}^{\sqrt{1-r^2\cos^2\theta}} r \, dz dr d\theta \\ &= 16 \int_0^{\frac{\pi}{4}} \int_0^1 r\sqrt{1-r^2\cos^2\theta} \, dr d\theta \\ &= 16 \int_0^{\frac{\pi}{4}} \left[-\frac{1}{2\cos^2\theta} \cdot \frac{2}{3}(1-r^2\cos^2\theta)^{\frac{3}{2}} \right]_0^1 d\theta \\ &= \frac{16}{3} \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - \frac{\sin^3\theta}{\cos^2\theta} \right) d\theta \\ &= \frac{16}{3} \left[\tan\theta - \frac{1}{\cos\theta} - \cos\theta \right]_0^{\frac{\pi}{4}} \\ &= \frac{16}{3}(3 - \frac{3}{\sqrt{2}}) \\ &= 8(2-\sqrt{2}).\end{aligned}$$

□

4. Let $\Omega \subset \mathbb{R}^2$ be the open subset bounded by the curve $x^2 - xy + 2y^2 = 1$. Express the integral $\int_\Omega xy \, dA$ as an integral over the unit disk in $\mathbb{R}^2$ centered at the origin.

Solution. Write $x^2 - xy + 2y^2 = (x - \frac{y}{2})^2 + (\frac{\sqrt{7}}{2}y)^2$. Then use the substitution $u = x - \frac{y}{2}$, $v = \frac{\sqrt{7}}{2}y$. □

5. Let $\Omega \subset \mathbb{R}^2$ be the open subset in the first quadrant bounded by $y = 0$, $y = x$, $xy = 1$ and $x^2 - y^2 = 1$. Evaluate the integral $\int_\Omega (x^2 + y^2) \, dA$ using the change of variables $u = xy$, $v = x^2 - y^2$.

Solution. Let $u = xy$, $v = x^2 - y^2$. We define function $g : (0,1)^2 \to \Omega$ as $g(u,v) = (x,y)$. Note that g is a diffeomorphism with inverse $h(x,y) = (xy, x^2 - y^2)$. By inverse function theorem, we have that

$$|\det(Dg)| = |\det(Dh)|^{-1} = \frac{1}{2}(x^2+y^2)^{-1} = \frac{1}{2\sqrt{v^2+4u^2}}.$$

By change of variables formula, we can conclude that

$$\int_\Omega (x^2+y^2)\, dA = \int_0^1 \int_0^1 \sqrt{v^2+4u^2}\, |\det(Dg)|\, dA = \frac{1}{2}.$$

□

6. Let $B^n(r)$ denote the closed ball of radius a in $\mathbb{R}^n$ centered at the origin.
   (a) Show that $\mathrm{Vol}(B^n(r)) = \lambda_n r^n$ for some positive constant λ_n.
   (b) Compute λ_1 and λ_2.
   (c) Compute λ_n in terms of λ_{n-2}.
   (d) Deduce a formula for λ_n for general n. *(Hint: consider two cases, according to whether n is even or odd.)*

Solution. (a) $\lambda_n = \mathrm{Vol}(B^n(1))$. The r^n arises from the change of variable $\vec{u} = r\vec{x}$, where $\vec{x} \in B^n(1)$.

(b) $\lambda_1 = 2$ and $\lambda_2 = \pi$.

(c) We introduce two methods for deriving the recursion formula.
Method 1: Let $B^n(o,1) = \left\{\vec{x} \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\right\}$ be the $n-$ball of radius 1 centered at the oringin in $\mathbb{R}^n$. Let $P = \{\vec{x} \in \mathbb{R}^n : x_3 = \cdots = x_n = 0\}$ be a $2-$plane. Let $S = B^n(1) \cap P$. It is straightforward to see that

$$S = \left\{\vec{x} \in \mathbb{R}^n : x_1^2 + x_2^2 < 1, x_3 = \cdots = x_n = 0\right\}.$$

Now $B^n(o,1)$ can be regard as the disjoint union of $(n-2)-$balls with centers $q \in S$ and radius $\sqrt{1-|q|^2}$, i.e.

$$B^n(1) = \sqcup_{q\in S} B^{n-2}(q, \sqrt{1-|q|^2}).$$

Therefore,

$$\begin{aligned}
\lambda_n &= \int_{B^n(o,1)} dV \\
&= \int_{x_1^2+x_2^2<1} \mathrm{Vol}(B^{n-2}(\sqrt{1-(x_1^2+x_2^2)}))\, dA \\
&= \lambda_{n-2} \int_{x_1^2+x_2^2<1} (\sqrt{1-(x_1^2+x_2^2)})^{n-2}\, dA \\
&= \lambda_{n-2} \int_0^{2\pi} \int_0^1 (\sqrt{1-r^2})^{n-2} r\, dr d\theta \\
&= \frac{2\pi}{n}\lambda_{n-2}.
\end{aligned}$$

Method 2: Let

$$I_n := \int_0^\pi \sin^n \theta \, d\theta.$$

Using integration by parts, we can derive the reduction formula for I_n:

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{for } n \geq 2, \quad I_0 = \pi, I_1 = 2.$$

Using spherical coordinates for $\mathbb{R}^n$, we have that

$$\begin{aligned}
\lambda_n &= \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^{n-1} (\sin^{n-2} \varphi_{n-1})(\sin^{n-3} \varphi_{n-2}) \cdots (\sin \varphi_2) \, d\rho d\varphi_1 d\varphi_2 \cdots d\varphi_{n-1} \\
&= \left(\int_0^\pi \sin^{n-2} \varphi_{n-1} \, d\varphi_{n-1} \right) \cdots \left(\int_0^\pi \sin \varphi_2 \, d\varphi_2 \right) \left(\int_0^{2\pi} d\varphi_1 \right) \left(\int_0^1 \rho^{n-1} \, d\rho \right) \\
&= \frac{2\pi}{n} \prod_{k=1}^{n-2} I_k \\
&= \frac{2\pi}{n} I_1 \prod_{k=2}^{n-2} \frac{k-1}{k} I_{k-2} \\
&= \frac{4\pi}{n(n-2)} \prod_{k=2}^{n-2} I_{k-2} \\
&= \frac{4\pi}{n(n-2)} I_0 \prod_{k=1}^{n-4} I_k \\
&= \frac{2\pi}{n} \left(\frac{2\pi}{n-2} \prod_{k=1}^{n-4} I_k \right) \\
&= \frac{2\pi}{n} \lambda_{n-2}.
\end{aligned}$$

(d) It follows directly from (c) that,

$$\lambda_n = \begin{cases} \frac{\pi}{k} \lambda_{2k-2} = \frac{\pi^k}{k!} & \text{if } n = 2k \\ \frac{2\pi}{2k+1} \lambda_{2k-1} = \frac{2(k!)(4\pi)^k}{(2k+1)!} & \text{if } n = 2k+1. \end{cases}$$

□

Suggested Exercises

1. Let $\Omega \subset \mathbb{R}^2$ be the region bounded below by $y = 1$ and above by $x^2 + y^2 = 4$. Evaluate

$$\int_\Omega (x^2 + y^2)^{-3/2} \, dA.$$

Solution. Using polar coordinates,

$$\begin{aligned}
\int_\Omega (x^2 + y^2)^{-3/2} \, dA &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc \theta}^2 r^{-3} \cdot r \, dr d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin \theta - \frac{1}{2} \right) \, d\theta \\
&= \sqrt{3} - \frac{\pi}{3}.
\end{aligned}$$

□

2. Find the area enclosed by the cardioid in $\mathbb{R}^2$ expressed in polar coordinates as $r = 1 + \cos\theta$.

Solution.

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi} \int_0^{1+\cos\theta} r \, dr d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) \, d\theta \\
&= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

□

3. Let $\Omega \subset \mathbb{R}^3$ be the region bounded below by the sphere $x^2+y^2+z^2 = 2z$ and above by the sphere $x^2+y^2+z^2 = 1$. Evaluate the integral

$$\int_\Omega \frac{z}{(x^2+y^2+z^2)^{3/2}} \, dV.$$

Solution. Using spherical coordinates,

$$\begin{aligned}
&\int_\Omega \frac{z}{(x^2+y^2+z^2)^{3/2}} \, dV \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \frac{\rho\cos\phi}{\rho^3}\rho^2 \sin\phi \, d\rho d\phi d\theta + \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{2\cos\phi} \frac{\rho\cos\phi}{\rho^3}\rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \frac{3\pi}{4} + \frac{\pi}{6} \\
&= \frac{11\pi}{12}.
\end{aligned}$$

□

4. Let $\Omega \subset \mathbb{R}^2$ be the open subset lying in the first quadrant and bounded by the hyperbolas $xy = 1$, $xy = 2$ and the lines $y = x$, $y = 4x$. Evaluate the integral $\int_\Omega x^2y^3 \, dA$.

Solution. We want $u = xy$ and $v = \frac{y}{x}$, i.e. $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$. Formally, we define $g : \Gamma := (1,2) \times (1,4) \to \Omega$ by $g(u,v) = (\sqrt{\frac{u}{v}}, \sqrt{uv})$. Then g is a bijective C^1 map with

$$Dg(u,v) = \begin{pmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix} \quad \text{and} \quad \det(Dg)(u,v) = \frac{1}{2v} > 0.$$

Hence, g is a diffeomorphism.

Note that both $\partial\Omega$ and $\partial\Gamma$ have measure zero.

By the change of variables theorem and Fubini's theorem, we have

$$\begin{aligned}\int_\Omega x^2y^3\, dA &= \int_1^2\int_1^4 (\sqrt{\frac{u}{v}})^2(\sqrt{uv})^3\cdot\frac{1}{2v}\, dvdu\\ &= \frac{1}{2}\int_1^2\int_1^4 u^{5/2}v^{-1/2}\, dvdu\\ &= \frac{2}{7}(8\sqrt{2}-1).\end{aligned}$$

□

5. Let $\Omega \subset \mathbb{R}^3$ be the open tetrahedron with vertices $(0,0,0)$, $(1,2,3)$, $(0,1,2)$ and $(-1,1,1)$. Evaluate the integral $\int_\Omega (x+2y-z)\, dV$.

Solution. Let $\Gamma = \{(u,v,w) : 0 < u+v+w < 1, u,v,w > 0\}$ and define $g : \Gamma \to \Omega$ by

$$g(u,v,w) = u(1,2,3) + v(0,1,2) + w(-1,1,1) = (u-w, 2u+v+w, 3u+2v+w).$$

Then g is a bijective C^1 map with

$$Dg(u,v,w) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \det(Dg)(u,v,w) = -2 < 0.$$

Hence, g is a diffeomorphism. Write

$$\Gamma = \{(u,v,w) : 0 < w < 1, 0 < v < 1-w, 0 < u < 1-v-w\}.$$

Note that both $\partial\Omega$ and $\partial\Gamma$ have measure zero.

By the change of variables theorem and Fubini's theorem, we have

$$\begin{aligned}\int_\Omega (x+2y-z)\, dV &= \int_\Gamma ((u-w) + 2(2u+v+w) - (3u+2v+w))\cdot|-2|\, dV\\ &= \int_0^1\int_0^{1-w}\int_0^{1-v-w} 4u\, dudvdw\\ &= \frac{1}{6}.\end{aligned}$$

□

6. Let $\Omega \subset \mathbb{R}^2$ be the open subset bounded by $x = 0$, $y = 0$ and $x + y = 1$. Evaluate the integral $\int_\Omega \cos\left(\frac{x-y}{x+y}\right)\, dA$. *(Hint: note that the integrand is un-defined at the origin.)*

Solution. First note that $f(x,y) := \cos\left(\frac{x-y}{x+y}\right)$ is a continuous function bounded on Ω.

Next we want $u = x + y$ and $v = x - y$, i.e. $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. Then

$$\begin{aligned}(x,y)\in\Omega &\iff 0 < x < 1 \text{ and } 0 < y < 1-x\\ &\iff 0 < u < 1 \text{ and } -u < v < u.\end{aligned}$$

Define $g : \Gamma := \{(u,v) : 0 < u < 1, -u < v < u\} \to \Omega$ by $g(u,v) = (\frac{1}{2}(u+v), \frac{1}{2}(u-v))$. Then g is a C^1 bijective map with

$$Dg(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \det(Dg)(u,v) = -\frac{1}{2} < 0.$$

Hence, g is a diffeomorphism.

Note that both $\partial\Omega$ and $\partial\Gamma$ have measure zero.

By the change of variables theorem and Fubini's theorem, we have

$$\begin{aligned} \int_\Omega \cos\left(\frac{x-y}{x+y}\right) dA &= \int_0^1 \int_{-u}^{u} \cos\left(\frac{v}{u}\right) \cdot \left|-\frac{1}{2}\right| \, dvdu \\ &= \int_0^1 u \sin(1) \, du = \frac{1}{2}\sin(1). \end{aligned}$$

□

7. Find the volume of the solid region $\Omega \subset \mathbb{R}^3$ bounded below by the surface $z = x^2 + 2y^2$ and above by the plane $z = 2x + 6y + 1$ by expressing it as an integral over the unit disk in $\mathbb{R}^2$ centered at the origin.

Solution. Note that $\Omega = \{(x,y,z) : (x,y) \in B, x^2 + 2y^2 < z < 2x + 6y + 1\}$, where B is the region in xy-plane bounded by the curve $x^2 + 2y^2 = 2x + 6y + 1$, i.e. $(x-1)^2 + 2(y - \frac{3}{2})^2 = \frac{13}{2}$. Then we use the substitution $u = \sqrt{\frac{2}{13}}(x-1)$, $v = \frac{2}{\sqrt{13}}(y - \frac{3}{2})$. □

8. Let $\Omega \subset \mathbb{R}^2$ be the open triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Evaluate the integral $\int_\Omega e^{(x-y)/(x+y)} \, dA$

   (a) using polar coordinates;

   (b) using the change of variables $u = x - y$, $v = x + y$.

Solution. (a) In polar coordinates, $\Omega = \{(r,\theta) : 0 < \theta < \frac{\pi}{2}, 0 < r < \frac{1}{\cos\theta + \sin\theta}\}$.

(b) Essentially the same as Q4 of "Suggested Excercise".

□

Challenging Exercises

1. (a) Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of one of the following types:

   (a) $g(e_i) = e_i$ for $i \neq j$, $g(e_j) = ae_j$

   (b) $g(e_i) = e_i$ for $i \neq j$, $g(e_j) = e_j + e_k$, $g(e_k) = e_k$ for $k \neq i, j$

   (c) $g(e_i) = e_j$, $g(e_j) = e_i$

   If U is a rectangle, show that the volume of $g(U)$ is $|\det(g)| \cdot \text{vol}(U)$.

   (b) Prove that $|\det(g)| \cdot \text{vol}(U)$ is the volume of $g(U)$ for any linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$. (Hint: If $\det(g) \neq 0$, then g is the composition of linear transformations of the type considered in (a).)

Solution. Check tutorial notes of 9 oct . □

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with measure zero $\partial\Omega$. Show that for any $\epsilon > 0$, there exists a compact subset $K \subset \Omega$ such that ∂K has measure zero and $\mathrm{Vol}(\Omega \setminus K) < \epsilon$.

Solution. Let $\varepsilon > 0$. Choose a rectangle $R \supset \Omega$. By the assumption, 1_Ω is integrable on R. So there is a partition $\mathcal{P}$ of R such that

$$U(1_\Omega, \mathcal{P}) - L(1_\Omega, \mathcal{P}) < \varepsilon.$$

Let $K = \bigcup_{Q \in \mathcal{P} : Q \subset \Omega} Q$. Clearly K is a subset of Ω. It is compact because it is a finite union of closed bounded rectangles. ∂K has measure zero because it consists of a finite unions of faces of rectangles. Finally, since $L(1_\Omega, \mathcal{P}) = L(1_K, \mathcal{P})$, we have

$$\mathrm{Vol}(\Omega \backslash K) \leq U(1_\Omega - 1_K, \mathcal{P}) \leq U(1_\Omega, \mathcal{P}) - L(1_K, \mathcal{P}) = U(1_\Omega, \mathcal{P}) - L(1_\Omega, \mathcal{P}) < \varepsilon.$$

□