MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to Problem Set 3

Notations: Throughout this problem set, we use R to denote a rectangle in $\mathbb{R}^n$. When we write $R = A \times B$, then we mean $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^k$ are rectangles with $n = m + k$.

Problems to hand in

1. Prove that if $A \subset \mathbb{R}^n$ is compact and has measure zero , then A has content zero .

   Solution. $\forall \varepsilon > 0$, there exist a sequence of open rectangles $R_1, R_2, \cdots$, such that

   $$A \subset \bigcup_{n=1}^{\infty} R_n$$

   and

   $$\sum_{n=1}^{\infty} Vol(R_n) < \varepsilon.$$

   then because A is compact , we can find finite many of such R_n such that

   $$A \subset \bigcup_{n=1}^{m} R'_n$$

   an at the same time $\sum Vol(R'_n) < \varepsilon$ then A is content zero . □

2. Define the *volume* of a subset $\Omega \subset \mathbb{R}^n$ by $Vol(\Omega) = \int_\Omega 1 dV$.

   (a) Let $A \subset \mathbb{R}^n$ be a content zero subset. Prove that A must be bounded. Moreover, show that ∂A has measure zero and $\text{Vol}(A) = 0$.

   (b) Let $B \subset \mathbb{R}^n$ be a bounded subset of measure zero. Suppose ∂B has measure zero. Prove that $\text{Vol}(B) = 0$.

   Solution. (a) A is bounded because it can be covered by a finite collection of closed bounded rectangles.

   $\bar{A}$ has content zero since $\bar{A}$ is the smallest closed set containing A and a finite union of closed sets is closed. So $\partial A \subset \bar{A}$ has content zero, hence measure zero.

   A finite collection of rectangles $\{R_i\}_{i=1}^k$ covering A will induce a partition $\mathcal{P}$ of a rectangle $R \supset A$ such that $U(\chi_A, \mathcal{P}) \leq \sum_{i=1}^k \text{Vol}(R_i)$, which can be made arbitrarily small.

   (b) As $\bar{B} = B \cup \partial B$, $\bar{B}$ also has measure zero. $\bar{B}$ is closed and bounded (since B is bounded), hence compact. So, by suggested exercise 3(c), $\bar{B}$ has content zero, and so does B. So $\text{Vol}(B) = 0$ by (a).

   □

3. Evaluate the following integrals:

   (a) $\int_R \frac{x}{x^2+y} \, dV$ where $R = [0, 1] \times [1, 3]$

(b) $\int_0^1 \int_{x^2}^{x} \frac{x}{1+y^2}\, dydx$

(c) $\int_0^1 \int_{\sqrt{y}}^{1} e^{y/x}\, dxdy$

Solution. (a) consider $[(x^2+3)\ln(x^2+3)]' = 2x\ln(x^2+3) + 2x$

$$\int_R \frac{x}{x^2+y}\, dV = \int_0^1 \int_1^3 \frac{x}{x^2+y}\, dydx = \int_0^1 x(\ln(x^2+3) - \ln(x^2+1))dx$$

$$= \frac{1}{2}(1^2+3)(\ln 1^2+3) - \frac{1}{2}(0^2+3)\ln(0^2+3) - \frac{1}{2}(1^2+1)\ln(1^2+1) + \frac{1}{2}(0^2+1)\ln(0^2+1)$$

$$= \frac{1}{2}(6\ln 2 - 3\ln 3).$$

(b)

$$\int_0^1 \int_{x^2}^{x} \frac{x}{1+y^2}\, dydx = \int_0^1 \int_y^{\sqrt{y}} \frac{x}{1+y^2}\, dxdy = \int_0^1 \frac{y-y^2}{1+y^2}dy = \int_0^1 \frac{1}{2+2y^2}d(y^2) - \int_0^1 (1-\frac{1}{1+y^2})dy$$

$$= \frac{1}{2}(\frac{1}{2}\ln 2 - 1 + \frac{\pi}{4}).$$

(c)

$$\int_0^1 \int_{\sqrt{y}}^{1} e^{y/x}\, dxdy = \int_0^1 \int_0^{x^2} e^{y/x}\, dydx = \int_0^1 x(e^x - e^0)dx = -\frac{1}{2} + [(x-1)e^x]_{x=1} - [(x-1)e^x]_{x=0} = \frac{1}{2}.$$

□

4. Let $\Omega \subset \mathbb{R}^3$ be the portion of the cube $[0,1]\times[0,1]\times[0,1]$ lying above the plane $y+z=1$ and below the plane $x+y+z=2$. Evaluate the integral $\int_\Omega x\, dV$.

Solution.

$$\int_\Omega x\, dV = \int_0^1 \int_0^1 \int_{1-y}^{2-x-y} xdzdydx - \int_0^1 \int_0^{1-x} \int_1^{2-y-x} xdzdydx$$

$$= \int_0^1 \int_0^1 x(1-x)dydx + \int_0^1 \int_0^{1-x} x(1-y-x)dydx = \frac{1}{6} - \frac{1}{24} = \frac{1}{8}.$$

□

5. Let $f : R = [0,1]\times[0,1] \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}, \\ 2x & \text{if } y \notin \mathbb{Q}. \end{cases}$$

(a) Prove that f is NOT integrable on R.

(b) Show that each iterated integral $\int_0^1 \int_0^1 f(x,y)\, dxdy$ and $\int_0^1 \overline{\int_0^1} f(x,y)\, dydx$ exist and compute their values.

Solution. (a) We want to show that f violates the integrability condition in challenging exercise 1 of Problem Set 1.

Let $\mathcal{P}_n := \{C^n_{i,j} := [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}] : 1 \leq i, j \leq n\}$ be a partition of R. Clearly, $\max_{i,j} \operatorname{diam}(C^n_{i,j}) \to 0$ as $n \to \infty$. However, note that

$$\begin{aligned} U(f, \mathcal{P}_{2n}) &= \frac{1}{2} + \sum_{j=1}^{2n} \sum_{i=n+1}^{2n} 2 \cdot \frac{i}{2n} (\frac{1}{2n})^2 \\ &= \frac{1}{2} + 2(2n) \frac{(n+1+2n)n}{2} (\frac{1}{2n})^3 \\ &= \frac{1}{2} + \frac{3n+1}{4n}, \end{aligned}$$

and

$$\begin{aligned} L(f, \mathcal{P}_{2n}) &= \sum_{j=1}^{2n} \sum_{i=1}^{n} 2 \cdot \frac{i-1}{2n} (\frac{1}{2n})^2 + \frac{1}{2} \\ &= 2(2n) \frac{(n-1)n}{2} (\frac{1}{2n})^3 + \frac{1}{2} \\ &= \frac{n-1}{4n} + \frac{1}{2}. \end{aligned}$$

Now, f is not integrable on R because

$$U(f, \mathcal{P}_{2n}) - L(f, \mathcal{P}_{2n}) \to \frac{1}{2} \neq 0.$$

(b) If $y \in \mathbb{Q}$, then $f(\,\cdot\,, y) = 1$ is integrable on $[0, 1]$ and

$$\int_0^1 f(x, y)\, dx = \int_0^1 1\, dx = 1.$$

If $y \notin \mathbb{Q}$, then $f(\,\cdot\,, y) = 2x$ is integrable on $[0, 1]$ and

$$\int_0^1 f(x, y)\, dx = \int_0^1 2x\, dx = 1.$$

Hence $y \mapsto \int_0^1 f(x, y)\, dx = 1$ is integrable on $[0, 1]$ and

$$\int_0^1 \int_0^1 f(x, y)\, dxdy = \int_0^1 1\, dy = 1.$$

By the density of $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$, it is easy to see that

$$\overline{\int_0^1} f(x, y)\, dy = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 2x & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

which is integrable over $[0, 1]$. Hence $\int_0^1 \overline{\int_0^1} f(x, y)\, dydx$ exists and

$$\int_0^1 \overline{\int_0^1} f(x, y)\, dydx = \int_0^{\frac{1}{2}} dx + \int_{\frac{1}{2}}^1 2x\, dx = \frac{5}{4}.$$

$\square$

Suggested Exercises

1. (a) Show that the subset $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ has measure zero.
   (b) Show that $\mathbb{Q}^c \cap [0,1]$ does not have measure zero in $\mathbb{R}$.

   Solution. (a) The set can be covered by the union of rectangles
   $$R_i = [-2^i, 2^i] \times \cdots \times [-2^i, 2^i] \times [-\varepsilon/2^{(i+1)n}, \varepsilon/2^{(i+1)n}], \quad i \in \mathbb{N},$$
   whose total volume is $\sum_{i=1}^{\infty} \mathrm{Vol}(R_i) = \sum_{i=1}^{\infty} (2 \cdot 2^i)^{n-1} \cdot \dfrac{2\varepsilon}{2^{(i+1)n}} = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$

   (b) If $\mathbb{Q}^c \cap [0,1]$ has measure zero, then $[0,1] = (\mathbb{Q} \cap [0,1]) \cup (\mathbb{Q}^c \cap [0,1])$ also has measure zero as $\mathbb{Q} \cap [0,1]$ has measure zero, hence volume zero, which is not true.

   □

2. Let $f : \Omega \to \mathbb{R}$ be a bounded continuous function defined on a bounded subset $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ has measure zero. Suppose Ω is path-connected, i.e. for any $p, q \in \Omega$, there exists a continuous path $\gamma(t) : [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Prove that there exists some $x_0 \in \Omega$ such that
   $$\int_\Omega f \, dV = f(x_0)\mathrm{Vol}(\Omega).$$

   Solution. Without loss of generality, we assume that f is non-constant and $\mathrm{Vol}(\Omega) \neq 0$. Then there exists $u \in \Omega$ such that
   $$m := \inf_{x\in\Omega} f(x) < f(u) < \inf_{x\in\Omega} f(x) =: M.$$
   Let $\varepsilon > 0$ be very small. By continuity, there is $\delta > 0$ such that
   $$m + \varepsilon < f(x) < M - \varepsilon \qquad \text{for all } x \in B_\delta(u) \cap \Omega.$$
   By considering $\int_\Omega f \, dV = \int_{\Omega\setminus B_\delta(u)} f \, dV + \int_{B_\delta(u)} f \, dV$, we have
   $$m\mathrm{Vol}(\Omega\setminus B_\delta(u)) + (m+\varepsilon)\mathrm{Vol}(B_\delta(u)) \leq \int_\Omega f \, dV \leq M\mathrm{Vol}(\Omega\setminus B_\delta(u)) + (M-\varepsilon)\mathrm{Vol}(B_\delta(u))$$
   $$m\mathrm{Vol}(\Omega) + \varepsilon\mathrm{Vol}(B_\delta(u)) \leq \int_\Omega f \, dV \leq M\mathrm{Vol}(\Omega) - \varepsilon\mathrm{Vol}(B_\delta(u)),$$
   and thus
   $$m + K\varepsilon \leq \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega f \, dV \leq M - K\varepsilon$$
   where $K = \frac{\mathrm{Vol}(B_\delta(u))}{\mathrm{Vol}(\Omega)} > 0$.
   By the definition of supremum and infimum, there exist $p_1, p_2 \in \Omega$ such that $f(p) > M - K\varepsilon$ and $f(q) < m + K\varepsilon$. Therefore,
   $$f(q) < \frac{1}{\mathrm{Vol}(\Omega))} \int_\Omega f \, dV < f(p).$$
   We want to apply the intermediate value theorem in 1-dimensional case. Since Ω is path-connected, there is a continuous path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Now $f \circ \gamma : [0,1] \to \mathbb{R}$ is a continuous function with $(f\circ\gamma)(0) = f(p)$ and $(f\circ\gamma)(1) = f(q)$. By intermediate value theorem, there exists $t_0 \in [0,1]$ such that $(f\circ\gamma)(t_0) = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega f \, dV$. Finally, $x_0 := \gamma(t_0) \in \Omega$ satisfies the desired property. □

3. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2+y^2=1$ and $x^2+z^2=1$.

Solution. Consider the part $x,y,z \geq 0$, when $x=h$,we have $y^2, z^2 \leq 1-h^2$, [1] so

$$\text{Volume of the region} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \; dydzdx = 4\int_{-1}^{1}(1-x^2)dx = \frac{16}{3}.$$

□

4. Find the volume of the region in $\mathbb{R}^3$ bounded below by the xy-plane, above by $z=y$, and on the sides by $y=4-x^2$.

Solution. Let $f(x,y)=y$, $\Omega=\{(x,y): 0\leq y\leq 4-x^2\}$. Then

$$\begin{aligned}\text{Volume of the region} &= \int_{\Omega} f \; dV\\ &= \int_{-2}^{2}\int_{0}^{4-x^2} y \; dydx\\ &= \int_{-2}^{2}\frac{1}{2}(4-x^2)^2 \; dx\\ &= \frac{256}{15}.\end{aligned}$$

□

5. Let $f:\Omega\to\mathbb{R}$ be a C^2 function [2] on an open subset $\Omega\subset\mathbb{R}^2$. Use Fubini's Theorem to prove that $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ everywhere in Ω.

Solution. Applying Fubini's Theorem and the fundamental theorem of calculus, one can show that

$$\int_{R'}\frac{\partial^2 f}{\partial x\partial y}\,dV = \int_{R'}\frac{\partial^2 f}{\partial y\partial x}\,dV,$$

and hence

$$\int_{R'}\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)\,dV = 0,$$

for any subretcangle $R'\subset\Omega$. The continuity of $\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}$ then implies that $\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}=0$ on Ω. □

6. Let $f:R=[a,b]\times[c,d]\to\mathbb{R}$ be a continuous function. Define another function $F:R\to\mathbb{R}$ such that

$$F(x,y) := \int_{[a,x]\times[c,y]} f \; dV.$$

Compute $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in the interior of R.

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[1] Know more interesting story about this , you can google Steinmetz solid .

[2] Recall that a function f is C^k if all the partial derivatives up to order k exist and are continuous.

Solution. By Fubini's Theorem and the fundamental theorem of calculus,

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial}{\partial x}\int_a^x \left(\int_c^y f\ dy\right) dx = \int_c^y f(x,y)\ dy,$$

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial}{\partial x}\int_c^y \left(\int_a^x f\ dx\right) dy = \int_a^x f(x,y)\ dx.$$

□

7. Let $f : R = [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial y}$ is continuous on R. Define $G : [c,d] \to \mathbb{R}$ such that

$$G(y) := \int_a^b f(x,y)\ dx.$$

(a) Show that G is continuous on $[c,d]$.

(b) Prove that G is differentiable on (c,d) and $G'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\ dx$.

Solution. (a) For $y, y_0 \in [c,d]$,

$$\begin{aligned}|G(y) - G(y_0)| &= \left|\int_a^b f(x,y)\ dx - \int_a^b f(x,y_0)\ dx\right| \\ &= \left|\int_a^b (f(x,y) - f(x,y_0))\ dx\right| \\ &\le \int_a^b |f(x,y) - f(x,y_0)|\ dx.\end{aligned}$$

Let $\varepsilon > 0$. Since f is continuous on the compact set R, it is uniformly continuous on R. Then there exists $\delta > 0$ such that

$$|f(x,y) - f(u,v)| < \frac{\varepsilon}{b-a} \quad \text{whenever } \|(x,y) - (u,v)\| < \delta.$$

Now, if $|y - y_0| < \delta$, then $\|(x,y) - (x,y_0)\| < \delta$, so that

$$|G(y) - G(y_0)| \le \int_a^b \frac{\varepsilon}{b-a}\ dx = \varepsilon.$$

Therefore G is uniformly continuous, hence continuous on $[c,d]$.

(b) For $y \in (c,d)$ and h small,

$$\left|\frac{G(y+h) - G(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\ dx\right| \le \int_a^b \left|\frac{f(x,y+h) - f(x,y)}{h} - \frac{\partial f}{\partial y}(x,y)\right|\ dx.$$

By Mean Value Theorem, there exists ξ between y and $y+h$ such that

$$\frac{f(x,y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}(x,\xi)$$

Let $\varepsilon > 0$. From the uniform continuity of $\frac{\partial f}{\partial y}$ on R, there exists $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| < \frac{\varepsilon}{b-a} \quad \text{whenever } \|(x,y) - (u,v)\| < \delta.$$

Now, if $0 < |h| < \delta$, we have $\|(x,\xi)-(x,y)\| \le |h| < \delta$, and so

$$\begin{aligned}\left|\frac{G(y+h)-G(y)}{h} - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx\right| &\le \int_a^b \left|\frac{\partial f}{\partial y}(x,\xi) - \frac{\partial f}{\partial y}(x,y)\right|\,dx \\ &\le \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon.\end{aligned}$$

Therefore, G is differentiable on (c,d) and $G'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$.

□

Challenging Exercises

1. The following exercise establishes the theorem that a bounded function $f : R \to \mathbb{R}$ is integrable if and only if f is continuous on R except on a set of measure zero. Let $f : R \to \mathbb{R}$ be a bounded function. For each $p \in R$ and $\delta > 0$, we define the *oscillation of f at p* as

$$o(f,p) = \lim_{\delta\to 0^+}\left(\sup_{x\in B_\delta(p)\cap R} f(x) - \inf_{x\in B_\delta(p)\cap R} f(x)\right).$$

   (a) Show that $o(f,p)$ is well-defined and non-negative. Prove that f is continuous at p if and only if $o(f,p) = 0$.

   (b) For any $\epsilon > 0$, let $D_\epsilon := \{p \in R \,:\, o(f,p) \ge \epsilon\}$. Show that D_ϵ is a closed subset and the set of discontinuities D of f is given as $D = \cup_{n=1}^\infty D_{1/n}$.

   (c) Suppose f is integrable on R. Prove that $D_{1/n}$ has content zero for any $n \in \mathbb{N}$. Hence, show that D has measure zero.

   (d) Suppose D has measure zero, prove that f is integrable on R.

Solution. (a) $o(f,p)$ is well-defined and non-negative because $o(f,p,\delta) := \sup_{x\in B_\delta(p)\cap R} f(x) - \inf_{x\in B_\delta(p)\cap R} f(x) \ge 0$ decreases as $\delta \to 0^+$.

Suppose f is continuous at p. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|f(x)-f(p)| < \varepsilon/2$ whenever $x \in B_\delta(p)$. Thus $|f(x)-f(y)| < \varepsilon$ whenever $x, y \in B_\delta(p)$, so that $o(f,p) \le \sup_{x\in B_\delta(p)\cap R} f(x) - \inf_{x\in B_\delta(p)\cap R} f(x) \le \varepsilon$. Therefore, $o(f,p) = 0$.

The converse is clear since

$$|f(x)-f(y)| \le \sup_{x\in B_\delta(p)\cap R} f(x) - \inf_{x\in B_\delta(p)\cap R} f(x) \quad \text{for } x, y \in B_\delta(p).$$

(b) $R\backslash D_\varepsilon$ is open because

$$\begin{aligned} p \in R\backslash D_\varepsilon &\implies \exists\delta > 0 \text{ s.t. } o(f,p,\delta) < \varepsilon \\ &\implies \forall y \in B_{\delta/2}(p), o(f,y,\delta/2) \le o(f,p,\delta) < \varepsilon \\ &\implies \forall y \in B_{\delta/2}(p), o(f,y) < \varepsilon \\ &\implies B_{\delta/2}(p) \subset R\backslash D_\varepsilon. \end{aligned}$$

By (a), $D = \{p \in R : o(f,p) > 0\} = \bigcup_{n=1}^\infty \{p \in R : o(f,p) \ge 1/n\} = \bigcup_{n=1}^\infty D_{1/n}$.

(c) Let $\varepsilon, \eta > 0$. Choose a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \eta$. Then

$$\varepsilon \sum_{\substack{Q \in \mathcal{P} \\ Q \cap D_\varepsilon \neq \emptyset}} \mathrm{Vol}(Q) \leq \sum_{\substack{Q \in \mathcal{P} \\ Q \cap D_\varepsilon \neq \emptyset}} [\sup_{x \in Q} f(x) - \inf_{y \in Q} f(y)] \mathrm{Vol}(Q) \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \eta$$

Since $\{Q \in \mathcal{P} : Q \cap D_\varepsilon \neq \emptyset\}$ covers D_ε, D_ε has content zero, hence measure zero. So $D = \bigcup_{n=1}^{\infty} D_{1/n}$ also has measure zero.

(d) Let $\varepsilon, \eta > 0$. D_ε has measure zero, hence content zero. So we can find a partition $\mathcal{P} = \{Q_i\}$ such that $\sum_{Q_i \cap D_\varepsilon \neq \emptyset} \mathrm{Vol}(Q_i) < \eta$. For any $x \in A := \bigcup_{Q_i \cap D_\varepsilon = \emptyset} Q_i$, we have $o(f, x) < \varepsilon$ and thus there exists $\delta_x > 0$ such that $f(p) - f(q) < \varepsilon$ for $p, q \in B_{\delta_x}(x)$. By the compactness of A, $A \subset B_{\delta_{x_1}/2}(x_1) \cup \cdots \cup B_{\delta_{x_m}/2}(x_m)$. By refining $\mathcal{P}$ if necessary, we have

$$\sup_{x \in Q_i} f(x) - \inf_{x \in Q_i} f(x) \leq \varepsilon \quad \text{ if } Q_i \cap D_\varepsilon = \emptyset.$$

Now $U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq (\eta + \varepsilon)\mathrm{Vol}(R)$. Therefore f is integrable on R.

□