MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to Problem Set 4

Problems to hand in

1. Calculate the line integral $\int_C f\,ds$ and $\int_C \mathbf{F}\cdot d\mathbf{r}$ where

   (a) $f(x,y,z) = y^2 + z - 3xy$, $\mathbf{F}(x,y,z) = (y^2, z, -3xy)$ and C is the line segment from $(1,0,1)$ to $(2,3,-1)$.

   (b) $f(x,y) = x + y$, $\mathbf{F}(x,y) = (-y^3, x^3)$ and C is the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ oriented counterclockwise.

Solution. (a) Solution

We are given:

$$f(x,y,z) = y^2 + z - 3xy, \quad \mathbf{F}(x,y,z) = (y^2, z, -3xy)$$

The curve C is the line segment from $(1,0,1)$ to $(2,3,-1)$.

The line segment from $(1,0,1)$ to $(2,3,-1)$ can be parametrized as:

$$\mathbf{r}(t) = (1+t, 3t, 1-2t), \quad 0 \le t \le 1$$

This gives:

$$x(t) = 1+t, \quad y(t) = 3t, \quad z(t) = 1-2t$$

The differential of the position vector is:

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = (1,3,-2)dt$$

The line integral of $f\,ds$ is given by:

$$\int_C f\,ds = \int_0^1 f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt$$

We compute the magnitude of $\frac{d\mathbf{r}}{dt}$:

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{1+9+4} = \sqrt{14}$$

Substitute $f(x,y,z) = y^2 + z - 3xy$ along the curve $\mathbf{r}(t)$:

$$f(\mathbf{r}(t)) = (3t)^2 + (1-2t) - 3(1+t)(3t)$$

$$f(\mathbf{r}(t)) = 9t^2 + 1 - 2t - 9t(1+t) = 9t^2 + 1 - 2t - 9t - 9t^2$$

$$f(\mathbf{r}(t)) = 1 - 11t$$

Thus, the line integral becomes:

$$\int_C f\,ds = \sqrt{14}\int_0^1 (1-11t)dt = \sqrt{14}\left[t - \frac{11}{2}t^2\right]_0^1$$

$$= \sqrt{14}\left(1 - \frac{11}{2}\right) = -\frac{9}{2}\sqrt{14}$$

The line integral of $\mathbf{F} \cdot d\mathbf{r}$ is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

Substituting $\mathbf{F}(x, y, z) = (y^2, z, -3xy)$ along the curve $\mathbf{r}(t)$:

$$\mathbf{F}(\mathbf{r}(t)) = (9t^2, 1 - 2t, -9t(1 + t))$$

Now, compute the dot product:

$$\mathbf{F}(\mathbf{r}(t)) \cdot (1, 3, -2) = 9t^2(1) + (1 - 2t)(3) + (-9t(1 + t))(-2)$$

$$= 9t^2 + 3 - 6t + 18t + 18t^2 = 27t^2 + 12t + 3$$

Thus, the line integral becomes:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (27t^2 + 12t + 3) dt$$

$$= \left[9t^3 + 6t^2 + 3t\right]_0^1 = 9 + 6 + 3 = 18$$

(b) Solution

We are given:

$$f(x, y) = x + y, \quad \mathbf{F}(x, y) = (-y^3, x^3)$$

The curve C is the square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$, oriented counterclockwise.

The curve C consists of four line segments:

- From $(0, 0)$ to $(1, 0)$: $\mathbf{r}_1(t) = (t, 0)$, $0 \leq t \leq 1$
- From $(1, 0)$ to $(1, 1)$: $\mathbf{r}_2(t) = (1, t)$, $0 \leq t \leq 1$
- From $(1, 1)$ to $(0, 1)$: $\mathbf{r}_3(t) = (1 - t, 1)$, $0 \leq t \leq 1$
- From $(0, 1)$ to $(0, 0)$: $\mathbf{r}_4(t) = (0, 1 - t)$, $0 \leq t \leq 1$

The line integral $\int_C f\, ds$ is computed by summing the integrals over each segment of the square.

$$\int_{C_1} f\, ds = \int_0^1 t\, dt = \frac{1}{2}$$

$$\int_{C_2} f\, ds = \int_0^1 (1 + t)\, dt = \frac{3}{2}$$

$$\int_{C_3} f\, ds = \int_0^1 (2 - t)\, dt = \frac{3}{2}$$

$$\int_{C_4} f\, ds = \int_0^1 (1 - t)\, dt = \frac{1}{2}$$

Thus, the total line integral is:

$$\int_C f\, ds = \frac{1}{2} + \frac{3}{2} + \frac{3}{2} + \frac{1}{2} = 4$$

in addition , by symmetry , $\int_C f\, ds = 2\int_C x\, ds$.

The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is computed by summing the integrals over each segment of the square.

Along the first segment $\mathbf{r}_1(t) = (t, 0)$:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$$

Along the second segment $\mathbf{r}_2(t) = (1, t)$:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$$

Along the third segment $\mathbf{r}_3(t) = (1 - t, 1)$:

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 1$$

Along the fourth segment $\mathbf{r}_4(t) = (0, 1 - t)$:

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 0$$

Thus, the total line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 1 + 1 + 0 = 2$$

□

On the other hand , we can give a solution for (b) in use of stokes equation :

Solution. Integral like

$\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (-y^3, x^3)$

The curve C is a square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$, oriented counterclockwise.

Stokes' Theorem states:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Where:

- C is the boundary of surface S (the region enclosed by C in the xy-plane),
- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$.

For $\mathbf{F}(x, y) = (-y^3, x^3)$, the curl in two dimensions is:

$$\nabla \times \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Where $F_1 = -y^3$ and $F_2 = x^3$. Thus:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x^3) = 3x^2$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(-y^3) = -3y^2$$

So the curl is:

$$\nabla \times \mathbf{F} = 3x^2 + 3y^2$$

The surface S is the unit square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. The area element $d\mathbf{S}$ is $dx\, dy$, and the integral becomes:

$$\iint_S (3x^2 + 3y^2)\, dx\, dy$$

We split this into two integrals:

$$\iint_S 3x^2\, dx\, dy + \iint_S 3y^2\, dx\, dy$$

$$\iint_S 3x^2\, dx\, dy = 3 \int_0^1 \int_0^1 x^2\, dx\, dy$$

First, integrate with respect to x:

$$\int_0^1 x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} = \frac{1}{3}$$

Thus:

$$\iint_S 3x^2\, dx\, dy = 3 \cdot \frac{1}{3} \cdot \int_0^1 dy = 1$$

$$\iint_S 3y^2\, dx\, dy = 3 \int_0^1 \int_0^1 y^2\, dx\, dy$$

First, integrate with respect to x:

$$\int_0^1 dx = 1$$

Thus:

$$\iint_S 3y^2\, dx\, dy = 3 \int_0^1 y^2\, dy = 3 \left[\frac{y^3}{3}\right]_0^1 = 3 \cdot \frac{1^3}{3} = 1$$

Combining both integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 1 + 1 = 2$$

In addition , Green's Theorem states:

$$\int_C P\, dx + Q\, dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

□

2. Let C be the curve of intersection of the upper hemisphere $x^2+y^2+z^2=4$, $z \geq 0$ and the cylinder $x^2+y^2=2x$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where $F(x,y,z)=(y,z,x)$.

Solution. If (x,y,z) lies on the required curve, then $(x-1)^2+y^2=1$ and $z=\sqrt{4-2x}$. Thus, the curve C can be parametrized by

$$\begin{aligned}\vec{r}(t) &= (1+\cos t, \sin t, \sqrt{2-2\cos t}) \\ &= (1+\cos t, \sin t, 2\sin(t/2)), \quad t \in [0,2\pi].\end{aligned}$$

Hence,

$$\begin{aligned}\int_C F \cdot d\vec{r} &= \int_0^{2\pi} (\sin t, 2\sin(t/2), 1+\cos t) \cdot (-\sin t, \cos t, \cos(t/2))\, dt \\ &= \int_0^{2\pi} \left(-\sin^2 t + 2\sin(t/2)\cos t + \cos(t/2) + \cos(t/2)\cos t\right)\, dt \\ &= -\pi - \frac{8}{3} + 0 + 0 \\ &= -\pi - \frac{8}{3}.\end{aligned}$$

□

3. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ is the vector field

$$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

and C is an arbitrary path from $(1,1)$ to $(2,2)$ not passing through the origin.

Solution. We first find a C^1 function $f : \mathbb{R}\backslash\{(0,0)\} \to \mathbb{R}^2$ such that $\nabla f = F$, i.e.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{x}{x^2+y^2}, \\ \dfrac{\partial f}{\partial y} = \dfrac{y}{x^2+y^2}. \end{cases}$$

The first equation implies

$$f(x,y) = \frac{1}{2}\log(x^2+y^2) + g(y),$$

for some C^1 function g. Substituting this into the second equation gives

$$g'(y) = 0 \implies g(y) = C, \text{ a constant.}$$

Choose $C=0$, we have

$$f(x,y) = \frac{1}{2}\log(x^2+y^2).$$

By the Fundamental Theorem of Calculus for line integrals, we have

$$\int_C F \cdot d\vec{r} = f(2,2) - f(1,1) = \log 2.$$

□

4. Determine which of the following vector field F is conservative on $\mathbb{R}^n$. For whose that are conservative, find a potential function f for it. For those that are not conservative, find a closed curve such that $\oint_C F \cdot d\vec{r} \neq 0$.

(a) $F(x, y) = (y^2, x^2)$;

(b) $F(x, y, z) = (y^2 z, 2xyz + \sin z, xy^2 + y \cos z)$.

Solution. (a) Since the compatibility condition is a necessary condition for conservative vector field, we want to check that the compatibility condition is not satisfied by F. Indeed,

$$\frac{\partial F_1}{\partial y} = 2y$$
$$\frac{\partial F_2}{\partial x} = 2x \neq 2y.$$

So F is not conservative.

Let C be a circle with parametrization $\vec{r}(t) = (1 + \cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\begin{aligned}\oint_C F \cdot d\vec{r} &= \int_0^{2\pi} (\sin^2 t, \cos^2 t + 2\cos t + 1) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} \left(-\sin^3 t + \cos^3 t + 2\cos^2 t + \cos t\right) dt \\ &= 2\int_0^{2\pi} \cos^2 t\, dt = 2\pi \neq 0.\end{aligned}$$

(b) Since $\mathbb{R}^3$ is simply connected, F is a conservative vector field if it satisfies the compatibility condition. Indeed,

$$\frac{\partial F_1}{\partial y} = 2yz, \qquad \frac{\partial F_2}{\partial x} = 2yz;$$
$$\frac{\partial F_2}{\partial z} = 2xy + \cos z, \qquad \frac{\partial F_3}{\partial y} = 2xy + \cos z;$$
$$\frac{\partial F_3}{\partial x} = y^2, \qquad \frac{\partial F_1}{\partial z} = y^2.$$

Next we compute the potential function $f : \mathbb{R}^3 \to \mathbb{R}$. Suppose

$$\frac{\partial f}{\partial x} = y^2 z$$
$$\frac{\partial f}{\partial y} = 2xyz + \sin z$$
$$\frac{\partial f}{\partial x} = xy^2 + y \cos z.$$

The first equation implies

$$f(x, y, z) = xy^2 z + g(y, z)$$

for some C^1 function g. Substituting this into the second equation gives

$$\frac{\partial g}{\partial y} = \sin z \implies g(y, z) = y \sin z + h(z).$$

Substituting this into the third equation gives

$$\frac{\partial h}{\partial z} = 0 \implies h(z) = C, \text{ a constant.}$$

Choose $C = 0$, we have

$$f(x, y, z) = xy^2 z + y \sin z.$$

□

5. Find the area of the region enclosed by the curve $x^{2/3} + y^{2/3} = 1$.

Solution. Parametrize the curve $C : x^{2/3} + y^{2/3} = 1$ by

$$\gamma(t) = (\cos^3 t, \sin^3 t), \quad 0 \le 0 \le 2\pi.$$

By Green's Theorem,

$$\begin{aligned} \text{Area} &= \int_C x\, dy \\ &= \int_0^{2\pi} \cos^3 t \cdot 3 \sin^2 t \cos t\, dt \\ &= 3 \int_0^{2\pi} \cos^4 t \sin^2 t\, dt \\ &= \frac{3\pi}{8}. \end{aligned}$$

□

Suggested Exercises

1. Calculate the line integral $\int_C F \cdot d\vec{r}$ where

(a) $F(x, y, z) = (z, x, y)$ and C is the line segment from $(0, 1, 2)$ to $(1, -1, 3)$.

(b) $F(x, y, z) = (y, 0, 0)$ where C is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane.

Solution. (a) Parametrization: $\gamma(t) = (1 - t)(0, 1, 2) + t(1, -1, 3)$, $t \in [0, 1]$.

(b) Along the intersection,

$$x^2 + y^2 + (-x - y)^2 = 2(x + \frac{y}{2})^2 + \frac{3}{2}y^2 = 1.$$

So we may let

$$\begin{cases} \sqrt{2}(x + \frac{y}{2}) = \cos t \\ \sqrt{\frac{3}{2}}y = \sin t, \end{cases} \quad t \in [0, 2\pi].$$

A parametrization of C can then be obtained by solving x, y, z in terms of t.

□

2. Calculate $\int_C F \cdot d\vec{r}$ where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$F(x,y,z) = \left(3x + y^2 + 2xz, 2xy + ze^{yz} + y, x^2 + ye^{yz} + ze^{z^2}\right)$$

and C is the parametrized curve $\gamma : [0,1] \to \mathbb{R}^3$ given by

$$\gamma(t) = \left(e^{t^7 \cos(2\pi t^{21})}, t^{17} + 4t^3 - 1, t^4 + (t - t^2)e^{\sin t}\right).$$

Solution. Check that F is conservative, so $\int_C F \cdot d\vec{r}$ depends on the end-points of C only. Solving $\nabla f = F$, a potential function f is given by

$$f(x,y,z) = (\frac{3}{2}x^2 + xy^2 + x^2 z) + (e^{yz} + \frac{1}{2}y^2) + \frac{1}{2}e^{z^2}.$$

By the Fundamental Theorem of Calculus for line integrals,

$$\int_C F \cdot d\vec{r} = f(\gamma(1)) - f(\gamma(0)).$$

□

3. Calculate the line integral $\int_C F \cdot d\vec{r}$ where

(a) $F(x,y) = (xy^3, 0)$ and C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise;

(b) $F(x,y) = (-y\sqrt{x^2+y^2}, x\sqrt{x^2+y^2})$ and C is the circle $x^2 + y^2 = 2x$ oriented counterclockwise.

Solution. (a) Parametrization: $\vec{r}(t) = (\cos t, \sin t), t \in [0, 2\pi]$.

$$\begin{aligned}\int_C F \cdot d\vec{r} &= \int_0^{2\pi} (\cos t \sin^3 t, 0) \cdot (-\sin t, \cos t)\, dt \\ &= -\int_0^{2\pi} \int \cos t \sin^4 t\, dt \\ &= -\frac{1}{5} \sin^5 t \Big|_0^{2\pi} \\ &= 0.\end{aligned}$$

(b) By Green's Theorem,

$$\int_C F \cdot d\vec{r} = \iint_D \left(\frac{\partial(x\sqrt{x^2+y^2})}{\partial x} - \frac{\partial(-y\sqrt{x^2+y^2})}{\partial y} \right) dA = \iint_D \left(3\sqrt{x^2+y^2}\right) dA,$$

where $D = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1\}$.

Using polar coordinates,

$$\begin{aligned}\int_C F \cdot d\vec{r} = \iint_D 3\sqrt{x^2+y^2}\, dA &= 3\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \cdot r\, dr d\theta \\ &= 3\int_{-\pi/2}^{\pi/2} \frac{8}{3} \cos^3 \theta\, d\theta \\ &= 8\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} \\ &= \frac{32}{3}.\end{aligned}$$

□

4. Let C be the circle $x^2 + y^2 = 2x$ oriented counterclockwise. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where

$$F(x,y) = \left(-y^2 + e^{x^2}, x + \sin(y^3)\right).$$

Solution. By Green's Theorem,

$$\int_C F \cdot d\vec{r} = \iint_D \left(\frac{\partial(x + \sin(y^3))}{\partial x} - \frac{\partial(-y^2 + e^{x^2})}{\partial y} \right) dA = \iint_D (1 + 2y)\ dA,$$

where $D = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \le 1\}$.

Using polar coordinates,

$$\begin{aligned}
\int_C F \cdot d\vec{r} = \iint_D (1+2y)\, dA &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (1 + 2r\sin\theta) \cdot r\, dr d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}(2\cos\theta)^2 + \frac{2}{3}(2\cos\theta)^3 \sin\theta \right) d\theta \\
&= \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta \\
&= \pi.
\end{aligned}$$

□

5. Find the area of the region enclosed by the curve

$$\gamma(t) = \left(\cos t + t\sin t, \sin t - t\cos t\right), \quad 0 \le t \le 2\pi$$

and the line segment from $(1, -2\pi)$ to $(1, 0)$.

Solution. Let Ω be the enclosed region, C be the curve γ, and L be the line segment from $(1, -2\pi)$ to $(1, 0)$.

By Green's Theorem,

$$\begin{aligned}
\text{Area} &= \int_{\partial\Omega} x\, dy = \int_C x\, dy + \int_L x\, dy \\
&= \int_0^{2\pi} (\cos t + t\sin t)(t\sin t)\, dt + \int_{-2\pi}^{0} (1)(1)\, dt \\
&= \int_0^{2\pi} \left(t\cos t \sin t + t^2 \sin^2 t\right)\, dt + 2\pi \\
&= \frac{1}{2}\int_0^{2\pi} \left(t\sin 2t + t^2 - t^2\cos 2t\right)\, dt + 2\pi.
\end{aligned}$$

Using integration by parts,

$$\int_0^{2\pi} t\sin 2t\, dt = -\frac{t}{2}\cos 2t\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \cos 2t\, dt = -\pi,$$

$$\int_0^{2\pi} t^2 \cos 2t \, dt = \frac{t^2}{2} \sin 2t \Big|_0^{2\pi} - \frac{1}{2} \int_0^{2\pi} 2t \sin 2t \, dt = \pi.$$

Hence,

$$\text{Area} = \frac{1}{2}\left(-\pi + \frac{(2\pi)^3}{3} - \pi\right) + 2\pi = \frac{4\pi^3}{3} + \pi.$$

□

Alternatively,

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_C -y\,dx + x\,dy + \frac{1}{2}\int_L -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} t^2\,dt + \frac{1}{2}\int_{-2\pi}^{0} (0+1)\,dt \\ &= \frac{4\pi^3}{3} + \pi.\end{aligned}$$

6. Let $0 < b < a$. Find the area under the curve $f(t) = (at - b\sin t, a - b\cos t)$, $0 \leq t \leq 2\pi$, above the x-axis.

Solution. Note that $f_1'(t) = a - b\cos t > 0$ for all $t \in [0, 2\pi]$, so the x-coordinate of the points on the curve increases as t increases. Denote the required region by R. Let L_1, L_2, L_3, L_4 be the left, bottom, right and top boundaries of R oriented counterclockwise.

By Green's Theorem,

$$\text{Area}(R) = -\int_{\partial R} y\,dx = -\sum_{i=1}^{4}\int_{L_i} y\,dx.$$

Note that

$$\int_{L_1} y\,dx = \int_{L_3} y\,dx = 0$$

since x is constant on L_1 and L_3; and

$$\int_{L_2} y\,dx$$

since $y = 0$ on L_2. Therefore,

$$\begin{aligned}\text{Area}(R) &= -\int_{L_4} y\,dx = \int_{-L_4} y\,dx \\ &= \int_0^{2\pi} (a - b\cos t)^2\,dt \\ &= \pi(2a^2 + b^2).\end{aligned}$$

□

7. Suppose C is a piecewise C^1 closed curve in $\mathbb{R}^2$ that intersects with itself finitely many times and does not pass through the origin. Show that the line integral

$$\frac{1}{2\pi}\int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

is always an integer. This is called the *winding number* of C around the origin.

Solution. Let $\gamma(t) = (x(t), y(t)) : [0,1] \to \mathbb{R}^2\backslash\{(0,0)\}$ be a piecewise C^1 parametrization of C. We claim that there is a piecewise C^1 function

$$\theta : [0,1] \to \mathbb{R} such that \gamma(t) = \|\gamma(t)\|(\cos\theta(t), \sin\theta(t)).$$

Define

$$\theta(t) = \mathrm{Arg}(\gamma(0)) + \int_0^t F(\gamma(s)) \cdot \gamma'(s)\, ds,$$

where $\mathrm{Arg}(\gamma(0))$ is the principal argument of $\gamma(0)$ within $[0, 2\pi)$, and $F : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2$ is the vector field

$$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Let $\Gamma(t) := \|\gamma(t)\|(\cos\theta(t), \sin\theta(t))$. Then $\Gamma(0) = \gamma(0)$, and

$$\begin{aligned}
\Gamma'(t) &= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}(\cos\theta(t), \sin\theta(t)) + \|\gamma(t)\|(-\sin\theta(t), \cos\theta(t))F(\gamma(t)) \cdot \gamma'(t) \\
&= \frac{1}{\|\gamma(t)\|}\big(x'(t)(x(t)\cos\theta(t) + y(t)\sin\theta(t)), y'(t)(x(t)\cos\theta(t) + y(t)\sin\theta(t))\big) \\
&= \big(x'(t), y'(t)\big) = \gamma'(t).
\end{aligned}$$

Thus $\gamma(t) = \Gamma(t) = \|\gamma(t)\|(\cos\theta(t), \sin\theta(t))$. Since C is a closed curve, we must have $\gamma(0) = \gamma(1)$, and hence $\theta(1) = \theta(0) + 2n\pi$ for some integer n.

Finally, direct computation gives

$$\begin{aligned}
\frac{1}{2\pi}\int_C -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy &= \frac{1}{2\pi}\int_0^1 -\frac{\|\gamma(t)\|^2\sin^2\theta(t)\theta'(t)}{\|\gamma(t)\|^2} + \frac{\|\gamma(t)\|^2\cos^2\theta(t)\theta'(t)}{\|\gamma(t)\|^2}\, dt \\
&= \frac{1}{2\pi}\int_0^1 \theta'(t)\, dt \\
&= \frac{1}{2\pi}(\theta(1) - \theta(0)) = n \in \mathbb{Z}.
\end{aligned}$$

□

Challenging Exercises

1. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{R}^n$ defined by

$$F(x_1, x_2, \cdots, x_n) = (f(r)x_1, f(r)x_2, \cdots, f(r)x_n)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function and $r := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.

(a) Suppose f is differentiable everywhere. Prove that for all $i, j = 1, \cdots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

on $\mathbb{R}^n \setminus \{\vec{0}\}$ where F_k is the k-th component function of the vector field F.

(b) Suppose f is continuous everywhere. Prove that F is a conservative vector field on $\mathbb{R}^n$.

Solution. (a) Note that, for $r > 0$,

$$\begin{aligned}\frac{\partial F_i}{\partial x_j} &= \frac{\partial(f(r)x_i)}{\partial x_j} \\ &= f(r)\delta^i_j + x_i \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_j} \\ &= f(r)\delta^i_j + \frac{\partial f}{\partial r}\frac{x_i x_j}{r}.\end{aligned}$$

Since this expression is symmetric in i and j, we must have $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ on $\mathbb{R}^n \backslash \{\vec{0}\}$.

(b) One can show that $g(x) = \int_0^r t f(t)\, dt$ if $\|x\| = r$ is the required potential function.

□