MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to Problem Set 5

Notations: All surfaces are contained inside $\mathbb{R}^3$ with rectangular coordinates (x, y, z). We use U to denote a bounded open subset of $\mathbb{R}^2$.

Problems to hand in

1. Let $a > 0$ be a fixed constant. Find the area of the portion of the cylinder $x^2 + y^2 = a^2$ lying above the xy-plane and below the plane $z = y$.

   Solution. Define $\varphi : (0, \pi) \times (0, 1) \to \mathbb{R}^3$ by

   $$\varphi(\theta, t) = (a\cos\theta, a\sin\theta, ta\sin\theta).$$

   $$Area = \int_0^\pi \int_0^1 a^2 \sin\theta dt d\theta = 2a^2$$

   $\square$

2. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$. Calculate $\int_S x^2 \, d\sigma$. *(Hint: make use of the symmetry)*

   Solution. By symmetry,

   $$\int_S x^2 \, d\sigma = \int_S y^2 \, d\sigma = \int_S z^2 \, d\sigma.$$

   Hence,

   $$\int_S x^2 \, d\sigma = \frac{1}{3}\int_S (x^2 + y^2 + z^2) \, d\sigma = \frac{1}{3}\int_S 1 \, d\sigma = \frac{4\pi}{3}.$$

   $\square$

3. Let S be the portion of the plane $x + 2y + 2z = 4$ lying in the first octant of $\mathbb{R}^3$, oriented with outward normal pointing upward. Find

   (a) the area of S,

   (b) $\int_S (x - y + 3z) \, d\sigma$,

   (c) $\int_S F \cdot \vec{n} \, d\sigma$ where $F(x, y, z) = (x, y, z)$.

   Solution. Note that the plane $x + 2y + 2z = 4$ intersects the coordinate axes at $(4, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$. Hence, a parametrization for S is given by $g : U \to \mathbb{R}^3$, where

   $$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 1, 0 < v < 1 - u\},$$

   $$\begin{aligned} g(u, v) &= u(4, 0, 0) + v(0, 2, 0) + (1 - u - v)(0, 0, 2) \\ &= (4u, 2v, 2 - 2u - 2v). \end{aligned}$$

Note that g is 1-1 and C^1. Moreover,

$$\begin{aligned} g_u &= (4, 0, -2) \\ g_v &= (0, 2, -2) \\ g_u \times g_v &= (4, 8, 8) \\ \|g_u \times g_v\| &= 12. \end{aligned}$$

(a)

$$\text{Area}(S) = \int_0^1 \int_0^{1-u} 12\, dv du = 6$$

(b)

$$\begin{aligned} \int_S (x - y + 3z)\, d\sigma &= \int_0^1 \int_0^{1-u} [4u - 2v + 3(1 - 2u - 2v)]\,(12)\, dv du \\ &= 16. \end{aligned}$$

(c)

$$\begin{aligned} \int_S F \cdot \vec{n}\, d\sigma &= \int_0^1 \int_0^{1-u} (4u, 2v, 2 - 2u - 2v) \cdot (4, 8, 8)\, dv du \\ &= 8. \end{aligned}$$

□

4. Let S be the portion of the helicoid given by the parametrization $r(u, v) : (0, 1) \times (0, 2\pi) \to \mathbb{R}$ by

$$r(u, v) = (u \cos v, u \sin v, v).$$

Suppose S is oriented by the upward pointing unit normal $\vec{n}$. Compute $\int_S F \cdot \vec{n}\, d\sigma$ where $F(x, y, z) = (0, x, 0)$.

Solution. $r_u \times r_v = (\sin v, -\cos v, u)$, $F(r(u, v)) \cdot (r_u \times r_v) = -u \cos^2 v$.

$$\int_S F \cdot \vec{n}\, d\sigma = \int_0^{2\pi} \int_0^1 -u \cos^2 v du dv = -\frac{\pi}{2}$$

□

Suggested Exercises

1. Find the area of the portion of the cone $z = \sqrt{2(x^2 + y^2)}$ lying beneath the plane $y + z = 1$.

Solution. Combining $z = \sqrt{2(x^2 + y^2)}$ and $z = 1 - y$, we have

$$\begin{aligned} 2(x^2 + y^2) &= (1 - y)^2 \\ 2x^2 + (y + 1)^2 &= 2. \end{aligned}$$

Thus, a parametrization for the required surface S is given by $g : U \to \mathbb{R}^3$, $g(u,v) = (u, v, f(u,v))$, where

$$U = \{(u,v) : 2u^2 + (v+1)^2 < 2\},$$

and

$$f(u,v) = \sqrt{2(u^2+v^2)}.$$

Note that g is 1-1 and C^1. Moreover,

$$\begin{aligned}\|g_u \times g_v\| &= \sqrt{1 + (f_u)^2 + (f_v)^2} \\ &= \sqrt{1 + \frac{2u^2}{u^2+v^2} + \frac{2v^2}{u^2+v^2}} \\ &= \sqrt{3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Area}(S) &= \iint_U \|g_u \times g_v\| \, dA \\ &= \int_{-1}^{1} \int_{-1-\sqrt{2-2u^2}}^{-1+\sqrt{2-2u^2}} \sqrt{3} \, dv du \\ &= 2\sqrt{3} \int_{-1}^{1} \sqrt{2-2u^2} \, du \\ &= 4\sqrt{6} \int_{0}^{1} \sqrt{1-u^2} \, du \\ &= \sqrt{6}\pi.\end{aligned}$$

(Alternative Method) Consider spherical coordinates:

$$\begin{aligned}x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi.\end{aligned}$$

Then $z = \sqrt{2(x^2+y^2)}$ implies $\phi = \arctan(\frac{1}{\sqrt{2}})$; $y + z = 1$ implies $r = \frac{\sqrt{3}}{\sin\theta + \sqrt{2}}$. Hence, a parametrization for the required surface S is given by $g : (0,1) \times (0, 2\pi) \to \mathbb{R}^3$,

$$\begin{aligned}g(t,\theta) &= \left(t\frac{\sqrt{3}}{\sin\theta+\sqrt{2}} \cdot \cos\theta \cdot \frac{1}{\sqrt{3}}, t\frac{\sqrt{3}}{\sin\theta+\sqrt{2}} \cdot \sin\theta \cdot \frac{1}{\sqrt{3}}, t\frac{\sqrt{3}}{\sin\theta+\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{3}} \right) \\ &= \left(\frac{t\cos\theta}{\sin\theta+\sqrt{2}}, \frac{t\sin\theta}{\sin\theta+\sqrt{2}}, \frac{\sqrt{2}t}{\sin\theta+\sqrt{2}} \right).\end{aligned}$$

Note that g is 1-1 and C^1. Moreover,

$$\begin{aligned}\frac{\partial g}{\partial t} &= \left(\frac{\cos\theta}{\sin\theta+\sqrt{2}}, \frac{\sin\theta}{\sin\theta+\sqrt{2}}, \frac{\sqrt{2}}{\sin\theta+\sqrt{2}} \right), \\ \frac{\partial g}{\partial \theta} &= \left(-\frac{t(1+\sqrt{2}\sin\theta)}{(\sin\theta+\sqrt{2})^2}, \frac{\sqrt{2}t\cos\theta}{(\sin\theta+\sqrt{2})^2}, -\frac{\sqrt{2}t\cos\theta}{(\sin\theta+\sqrt{2})^2} \right) \\ \frac{\partial g}{\partial t} \times \frac{\partial g}{\partial \theta} &= \left(-\frac{\sqrt{2}t\cos\theta}{(\sin\theta+\sqrt{2})^2}, -\frac{\sqrt{2}t\sin\theta}{(\sin\theta+\sqrt{2})^2}, -\frac{t}{(\sin\theta+\sqrt{2})^2} \right).\end{aligned}$$

So,

$$\left\| \frac{\partial g}{\partial t} \times \frac{\partial g}{\partial \theta} \right\| = \frac{\sqrt{3}t}{(\sin\theta + \sqrt{2})^2}.$$

Therefore,

$$\begin{aligned}
\text{Area}(S) &= \iint_{(0,1)\times(0,2\pi)} \left\| \frac{\partial g}{\partial t} \times \frac{\partial g}{\partial \theta} \right\| dA \\
&= \int_0^{2\pi} \int_0^1 \frac{\sqrt{3}t}{(\sin\theta + \sqrt{2})^2} \, dt d\theta \\
&= \frac{\sqrt{3}}{2} \int_0^{2\pi} \frac{1}{(\sin\theta + \sqrt{2})^2} \, d\theta \\
&= \sqrt{6}\pi.
\end{aligned}$$

□

2. Find the area of the portion of the cylinder $x^2 + y^2 = 2y$ lying inside the sphere $x^2 + y^2 + z^2 = 4$.

Solution. In cylindrical coordinates, the cylinder $x^2 + y^2 = 2y$ is represented by $r = 2\sin\theta$, $\theta \in (0, \pi)$; the sphere $x^2 + y^2 + z^2 = 4$ is represented by $z^2 = 4 - r^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$. Thus, a parametrization for the given surface is $g : U \to \mathbb{R}^3$, where

$$U := \{(\theta, z) : 0 < \theta < \pi, -2|\cos\theta| < z < 2|\cos\theta|\},$$

$$\begin{aligned}
g(\theta, z) &= ((2\sin\theta)\cos\theta, (2\sin\theta)\sin\theta, z) \\
&= (\sin 2\theta, 1 - \cos 2\theta, z).
\end{aligned}$$

Hence,

$$|g_\theta \times g_z| = 2,$$

and

$$\text{Area} = \int_0^\pi \int_{-2|\cos\theta|}^{2|\cos\theta|} 2 \, dz d\theta = \int_0^\pi 8|\cos\theta| \, d\theta = 16.$$

□

3. Find the flux of the vector field $F(x, y, z) = (x^2, y^2, z^2)$ outward across the given surface S (all oriented with outward pointing normal pointing away from the origin, unless otherwise specified):

   (a) S is the sphere of radius a centered at the origin.

   (b) S is the upper hemisphere of radius a centered at the origin.

   (c) S is the cone $z = \sqrt{x^2 + y^2}$, $0 < z < 1$, with outward pointing normal having a negative z-component.

   (d) S is the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$.

   (e) S is the cylinder $x^2 + y^2 = a^2$, $0 \le z \le h$, along with the disks $x^2 + y^2 \le a^2$, $z = 0$ and $z = h$.

Solution. (a) Define $g : U := (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ by

$$g(u, v) = (a \sin u \cos v, a \sin u \sin v, a \cos u).$$

Then,

$$\begin{aligned}
\text{Flux of } F \text{ across } S &= \iint_U (F \circ g) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right) dA \\
&= \int_0^{\pi} \int_0^{2\pi} a^4(\sin^4 u \cos^3 v + \sin^4 \sin^3 v + \sin u \cos^3 u) \, dv du \\
&= 0.
\end{aligned}$$

(b) By (a),

$$\begin{aligned}
\text{Flux of } F \text{ across } S &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} a^4(\sin^4 u \cos^3 v + \sin^4 \sin^3 v + \sin u \cos^3 u) \, dv du \\
&= \frac{\pi a^4}{2}.
\end{aligned}$$

(c) Define $g : U := (0, 2\pi) \times (0, 1) \to \mathbb{R}^3$ by

$$g(\theta, z) = (z \cos \theta, z \sin \theta, z).$$

Then,

$$\begin{aligned}
\text{Flux of } F \text{ across } S &= \int_0^1 \int_0^{2\pi} z^3(\cos^3 + \sin^3 \theta - 1) \, d\theta dz \\
&= -\frac{\pi}{2}.
\end{aligned}$$

(d) Define $g : U := (0, 2\pi) \times (0, h) \to \mathbb{R}^3$ by

$$g(\theta, z) = (a \cos \theta, a \sin \theta, z).$$

Then,

$$\begin{aligned}
\text{Flux of } F \text{ across } S &= \int_0^{2\pi} \int_0^h a^3(\cos^3 + \sin^3 \theta) \, dz d\theta \\
&= 0.
\end{aligned}$$

(e) Disk on $z = 0$: Define $g_1 : (0, a) \times (0, 2\pi) \to \mathbb{R}^3$ by

$$g_1(r, \theta) = (r \cos \theta, r \sin \theta, 0).$$

Then,

$$\text{Flux} = \int_0^a \int_0^{2\pi} 0 \, d\theta dr = 0.$$

Disk on $z = h$: Define $g_2 : (0, a) \times (0, 2\pi) \to \mathbb{R}^3$ by

$$g_2(r, \theta) = (r \cos \theta, r \sin \theta, h).$$

Then,

$$\text{Flux} = \int_0^{2\pi} \int_0^a rh \, dr d\theta = \pi h^2 a^2.$$

So total flux of F across S is $0 + 0 + \pi h^2 a^2 = \pi h^2 a^2$.

□

4. Calculate the flux of the vector field $F(x, y, z) = (xz, yz, x^2 + y^2)$ outward across the surface of the paraboloid S given by $z = 4 - x^2 - y^2$, $z \geq 0$ (with outward pointing normal having positive z-component).

Solution. A parametrization for the surface $S \backslash \{z = 0\}$ is $g : U := \{(u, v) : u^2 + v^2 < 4\} \to \mathbb{R}^3$,

$$g(u, v) = (u, v, 4 - u^2 - v^2).$$

Then

$$g_u = (1, 0, -2u), \qquad g_v = (0, 1, -2v),$$

so that

$$g_u \times g_v = (2u, 2v, 1),$$

which is an upward pointing normal for S.

Therefore,

$$\begin{aligned}
\text{Flux} &= \iint_U (F \circ g) \cdot (g_u \times g_v)\, dA \\
&= \iint_U (x^2 + y^2)(9 - 2(x^2 + y^2))\, dA \\
&= \int_0^{2\pi} \int_0^2 (9r^2 - 2r^5) r\, dr d\theta \\
&= \frac{88\pi}{3}.
\end{aligned}$$

□

5. Compute $\int_S F \cdot \vec{n}\ d\sigma$ where $F(x, y, z) = (x, y, z)$ for each of the following surfaces in $\mathbb{R}^3$ (all oriented with the outward pointing unit normal pointing away from the origin):

(a) the sphere of radius a centered at the origin,

(b) the cylinder $x^2 + y^2 = a^2$, $-h \leq z \leq h$,

(c) the cylinder $x^2 + y^2 = a^2$, $-h \leq z \leq h$, together with the two disks $x^2 + y^2 \leq a^2$, $z = \pm h$,

(d) the cube with vertices at $(\pm 1, \pm 1, \pm 1)$.

Solution. (a)

$$\int_0^{\pi} \int_0^{2\pi} (a \sin u \cos v, a \sin u \sin v, a \cos u) \cdot a^2 \sin u (\sin u \cos v, \sin u \sin v, \cos u)\, dv du$$

(b)

$$\int_0^{2\pi} \int_{-h}^{h} (a \cos\theta, a \sin\theta, z) \cdot (a \cos\theta, a \sin\theta, 0)\, dz d\theta.$$

(c) Disk on $z = -h$:

$$\int_0^{2\pi} \int_0^{a} (r \cos\theta, r \sin\theta, -h) \cdot (0, 0, -r)\, dr d\theta.$$

Disk on $z = h$:

$$\int_0^{2\pi} \int_0^{a} (r \cos\theta, r \sin\theta, h) \cdot (0, 0, r)\, dr d\theta.$$

(d) By symmetry, it suffices to look at the side of the cube lying on the plane $x = 1$. Define $\varphi : (0,1)^2 \to \mathbb{R}^3$ by $\varphi(y,z) = (1,y,z)$. Then $\varphi_y \times \varphi_z = (1,0,0)$.

$$\text{Flux} = 6\int_0^1\int_0^1 (1,y,z)\cdot(1,0,0)\,dydz.$$

□

6. Repeat the question above for the vector field $F(x,y,z) = (x^2+y^2+z^2)^{-3/2}(x,y,z)$.

Solution. (a)

$$\int_0^\pi \frac{1}{a^3}\int_0^{2\pi} (a\sin u\cos v, a\sin u\sin v, a\cos u)\cdot a^2\sin u(\sin u\cos v, \sin u\sin v, \cos u)\,dvdu$$

(b)

$$\int_0^{2\pi} \frac{1}{(a^2+z^2)^{3/2}}\int_{-h}^{h} (a\cos\theta, a\sin\theta, z)\cdot(a\cos\theta, a\sin\theta, 0)\,dzd\theta.$$

(c) Disk on $z = -h$:

$$\int_0^{2\pi}\int_0^a \frac{1}{(r^2+z^2)^{3/2}}(r\cos\theta, r\sin\theta, -h)\cdot(0,0,-r)\,drd\theta.$$

Disk on $z = h$:

$$\int_0^{2\pi}\int_0^a \frac{1}{(r^2+z^2)^{3/2}}(r\cos\theta, r\sin\theta, h)\cdot(0,0,r)\,drd\theta.$$

(d) By symmetry,

$$\text{Flux} = 6\int_0^1\int_0^1 \frac{1}{(1+y^2+z^2)^{3/2}}(1,y,z)\cdot(1,0,0)\,dydz.$$

□

7. Prove that the area of a graphical surface S given by $z = f(x,y)$, where $f : U \to \mathbb{R}$ is a C^1 function, is given by

$$\text{Area}(S) = \iint_U \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA.$$

Solution. A parametrization of S is given by $\varphi : U \to \mathbb{R}^3$,

$$\varphi(x,y) = (x,y,f(x,y)).$$

Then $\varphi_x \times \varphi_y = (-f_x, -f_y, 1)$, and $\text{Area}(S) = \iint_U \sqrt{1+(f_x)^2+(f_y)^2}\,dA.$ □

Challenging Exercises

1. Let $\alpha, \beta, f : [0,1] \to \mathbb{R}$ be C^1 functions with $f(t) > 0$ for all $t \in [0,1]$. Suppose that S is a surface in $\mathbb{R}^3$ whose intersection with the plane $z = t$ is the circle

$$(x-\alpha(t))^2 + (y-\beta(t))^2 = (f(t))^2, \qquad z = t$$

for each $t \in [0,1]$ and is empty for $t \notin [0,1]$.

(a) Set up an integral for the area of S.

(b) Evaluate the integral in (a) when α and β are constant functions and $f(t) = (1+t)^{1/2}$.

(c) What form does the integral take when f is constant and $\alpha(t) = 0$ and $\beta(t) = at$ where a is a constant?

Solution. Define $\varphi : (0,1) \times (0, 2\pi) \to \mathbb{R}^3$ by

$$\varphi(t,\theta) = (\alpha(t) + f(t)\cos\theta, \beta(t) + f(t)\sin\theta, t).$$

□