MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to midterm

1. Find the area of the region cut from the first quadrant by the curve $r = 1 + \sin(\theta)$ given in polar coordinates.

Solution. The area enclosed by a polar curve $r = f(\theta)$ between two angles θ_1 and θ_2 is given by the formula:

$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.$$

For the curve $r = 1 + \sin(\theta)$, the area in the first quadrant corresponds to the range of θ from 0 to $\frac{\pi}{2}$, since the first quadrant is bounded by these angles.

Thus, the area is:

$$A = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \sin(\theta))^2 \, d\theta.$$

First, we expand $(1 + \sin(\theta))^2$:

$$(1 + \sin(\theta))^2 = 1 + 2\sin(\theta) + \sin^2(\theta).$$

Now, substitute this into the integral:

$$A = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(1 + 2\sin(\theta) + \sin^2(\theta)\right) \, d\theta.$$

We can split the integral into three parts:

$$A = \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} 1 \, d\theta + 2 \int_0^{\frac{\pi}{2}} \sin(\theta) \, d\theta + \int_0^{\frac{\pi}{2}} \sin^2(\theta) \, d\theta \right].$$

$$\int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}} \sin(\theta) \, d\theta = [-\cos(\theta)]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = 1.$$

Thus, $2 \int_0^{\frac{\pi}{2}} \sin(\theta) \, d\theta = 2 \times 1 = 2$.

Using the identity $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$, we have:

$$\int_0^{\frac{\pi}{2}} \sin^2(\theta) \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos(2\theta)) \, d\theta.$$

Now, evaluate each part:

$$\int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{2}, \quad \int_0^{\frac{\pi}{2}} \cos(2\theta) \, d\theta = \left[\frac{\sin(2\theta)}{2} \right]_0^{\frac{\pi}{2}} = 0.$$

Thus:

$$\int_0^{\frac{\pi}{2}} \sin^2(\theta) \, d\theta = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}.$$

Now, summing all parts:

$$A = \frac{1}{2}\left(\frac{\pi}{2} + 2 + \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{2\pi}{4} + 2 + \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{3\pi}{4} + 2\right).$$

Thus, the area is:

$$A = \frac{3\pi}{8} + 1.$$

□

2. Let D be the region in the first octant bounded by the coordinate planes, the plane $2y + z = 2$, and the surface $x = 1 - y^2$.

   (a) (10 points) Sketch the region D.

   (b) (10 points) Using a triple integral, find the volume of D.

Solution. Part (a): Sketch the region D

The region D is bounded by: - The coordinate planes $x = 0$, $y = 0$, and $z = 0$. - The plane $2y + z = 2$, which can be rewritten as $z = 2 - 2y$. - The surface $x = 1 - y^2$, which is a downward opening parabola in the x-direction.

To sketch the region: 1. The plane $2y + z = 2$ intersects the z-axis at $(0, 0, 2)$ and the y-axis at $(0, 1, 0)$. 2. The surface $x = 1 - y^2$ defines a parabola for each fixed z, with the maximum value of $x = 1$ when $y = 0$.

Thus, the region D is within the first octant, bounded by the plane and the paraboloid.

Part (b): Find the volume using a triple integral

We can compute the volume of region D using a triple integral. The limits of integration are determined by the bounds of the region.

1. **x-bound**: The surface is given by $x = 1 - y^2$, so for a fixed y, x ranges from 0 to $1 - y^2$.

2. **y-bound**: From the plane $2y + z = 2$, we know the maximum value of y occurs when $z = 0$, which gives $y = 1$. Therefore, y ranges from 0 to 1.

3. **z-bound**: For each fixed y, z ranges from 0 to $2 - 2y$ (from the plane equation).

The volume integral is set up as:

$$V = \int_0^1 \int_0^{2-2y} \int_0^{1-y^2} dx\, dz\, dy.$$

First, integrate with respect to x:

$$\int_0^{1-y^2} dx = 1 - y^2.$$

Now the volume becomes:

$$V = \int_0^1 \int_0^{2-2y} (1 - y^2) dz\, dy.$$

Next, integrate with respect to z:

$$\int_0^{2-2y} dz = 2 - 2y.$$

Thus, the volume is:

$$V = \int_0^1 (2 - 2y)(1 - y^2)\, dy.$$

Expand the integrand:

$$(2 - 2y)(1 - y^2) = 2(1 - y^2) - 2y(1 - y^2) = 2 - 2y^2 - 2y + 2y^3.$$

Now, integrate term by term:

$$V = \int_0^1 (2 - 2y^2 - 2y + 2y^3)\, dy.$$

The integrals of each term are:

$$\int_0^1 2\, dy = 2, \quad \int_0^1 -2y^2\, dy = -\frac{2}{3}, \quad \int_0^1 -2y\, dy = -1, \quad \int_0^1 2y^3\, dy = \frac{1}{2}.$$

Thus, the total volume is:

$$V = 2 - \frac{2}{3} - 1 + \frac{1}{2} = \frac{12}{6} - \frac{4}{6} - \frac{6}{6} + \frac{3}{6} = \frac{5}{6}.$$

Therefore, the volume of the region D is:

$$\boxed{\frac{5}{6}}.$$

□

3. (a) (10 points) Using cylindrical coordinates, find the volume of the solid bounded by the graphs of

$$z = 4 - 4(x^2 + y^2)$$

and

$$z = (x^2 + y^2)^2 - 1.$$

(b) (10 points) Using spherical coordinates, find the volume of the solid bounded from above by the graph $z = \sqrt{x^2 + y^2}$ and below by the surface $x^2 + y^2 + z^2 = 2z$.

Solution. Part (a): Volume using cylindrical coordinates

First, we convert the given equations into cylindrical coordinates. Recall that in cylindrical coordinates, $x^2 + y^2 = r^2$, and we use $z = (x^2 + y^2)^2 - 1$.

1. The first surface is given by:

$$z = 4 - 4(x^2 + y^2) = 4 - 4r^2.$$

2. The second surface is given by:

$$z = (x^2 + y^2)^2 - 1 = r^4 - 1.$$

We are tasked with finding the volume between these two surfaces. The z-bounds are from $z = r^4 - 1$ to $z = 4 - 4r^2$, and the r-bounds are determined by the intersection of these two surfaces.

To find the intersection, set the two equations equal:

$$4 - 4r^2 = r^4 - 1.$$

Rearranging:

$$r^4 - 4r^2 + 5 = 0.$$

This is a quadratic equation in r^2.

The limits of r are from 0 to 1 (the domain where the surfaces are valid). The volume integral in cylindrical coordinates is:

$$V = \int_0^{2\pi} \int_0^1 [(4 - 4r^2) - (r^4 - 1)]r \, dr \, d\theta.$$

Now the volume becomes:

$$V = 2\pi \int_0^1 (5 - 4r^2 - r^4)r \, dr.$$

Integrate with respect to r:

$$\int_0^1 (5r - 4r^3 - r^5) \, dr = \left[\frac{5r^2}{2} - r^4 - \frac{r^6}{6}\right]_0^1 = \frac{5}{2} - 1 - \frac{1}{6} = \frac{4}{3}.$$

Finally, integrate with respect to θ:

$$V = \int_0^{2\pi} \frac{4}{3} \, d\theta = \frac{4}{3} \times 2\pi = \frac{8\pi}{3}.$$

Thus, the volume is:

$$\boxed{\frac{8\pi}{3}}.$$

Part (b): Volume using spherical coordinates

The surface $x^2 + y^2 + z^2 = 2z$ can be rewritten as:

$$x^2 + y^2 + (z - 1)^2 = 1,$$

which represents a sphere with center $(0, 0, 1)$ and radius 1.

$$\begin{cases} x = \sin\phi\cos\theta & \rho \geq 0 \\ y = \rho\sin\phi\sin\theta & 0 \leq \theta \leq 2\pi \\ z = \rho\cos\phi & 0 \leq \phi\pi \end{cases}$$

The second surface is $z = \sqrt{x^2 + y^2}$, or in cylindrical coordinates, $z = \rho\cos(\phi) = \rho\sin(\phi)$.

In spherical coordinates, the equation for the sphere is $\rho^2 = 2\rho\cos(\phi)$.

The volume integral is:

$$V = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos(\phi)} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.$$

First, integrate with respect to ρ:

$$\int_0^{2\cos(\phi)} \rho^2\, d\rho = \left[\frac{\rho^3}{3}\right]_0^{2\cos(\phi)} = \frac{(2\cos(\phi))^3}{3} = \frac{8\cos^3(\phi)}{3}.$$

Now the volume becomes:

$$V = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{8\cos^3(\phi)}{3} \sin(\phi)\, d\phi\, d\theta.$$

Integrate with respect to ϕ:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\, d\phi = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3(\phi)\, d\cos\phi.$$

Thus, the volume is:

$$V = 2\pi\frac{8}{3} \times \frac{1}{4} \times (\frac{1}{\sqrt{2}})^4 = \frac{\pi}{3}.$$

Therefore, the volume of the solid is:

$$\boxed{\frac{\pi}{3}}.$$

□

4. (a) (5 points) Find the Jacobian (determinant) $\frac{\partial(x,y)}{\partial(u,v)}$ if $u = x + 2y$ and $v = y - x$, without solving (x, y) as functions of (u, v).
   (b) (10 points) Using change of variable formula, evaluate

$$\int_0^{\frac{\pi}{3}} \int_y^{\frac{\pi}{2}-2y} (x + 2y)e^{y-x}\, dx\, dy.$$

Solution. Part (a): Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

We are given the transformations:

$$u = x + 2y, \quad v = y - x.$$

We want to compute the Jacobian, which is given by:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}.$$

First, calculate $\frac{\partial(u,v)}{\partial(x,y)}$, that is, the determinant of the following matrix:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

The determinant is:

$$\det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = (1)(1) - (-1)(2) = 1 + 2 = 3.$$

Thus, the Jacobian is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}.$$

Part (b):

$$I = \int_0^{\frac{2}{3}} \int_y^{2-2y} (x+2y)e^{y-x}\,dx\,dy.$$

let

$$\begin{cases} u = x + 2y \\ v = y - x \end{cases}$$

$$I = \int_0^2 \int_{-u}^0 ue^v \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{3}\int_0^2 u(e^u - 1)\,du = \frac{1}{3}[2e^2 - \int_0^2 e^u\,du] - \frac{1}{3}\int_0^2 u\,du$$

$$I = \frac{1}{3}(e^2 - 1)$$

□

5. • (10 points) Consider the function:

$$f(x,y) = \begin{cases} 2, & \text{if both } x \text{ and } y \text{ are rational,} \\ -1, & \text{otherwise.} \end{cases}$$

Determine if f is Riemann integrable over the unit square $[0,1] \times [0,1]$. If it is, find the value of the integral. If not, explain why.

Solution. The function f is not Riemann integrable over the unit square. To see this, consider any partition of the unit square . Any rectangle that contains both rational and irrational points will have an upper Darboux sum of 2 and a lower Darboux sum of -1 for f over that rectangle.

Since the rational numbers are dense in the reals, every open set contains both rational and irrational numbers. Thus, no matter how to give the partition, the upper and lower Darboux sums will not converge to the same value. Therefore, f is not Riemann integrable over the unit square. □

• (10 points) Consider the function:

$$g(x,y) = \begin{cases} \sin\left(\frac{1}{y-x}\right), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Determine if g is Riemann integrable over the unit square $[0,1] \times [0,1]$. If it is, find the value of the integral. If not, explain why.

Solution. Hint : The function g is Riemann integrable over the unit square. Note that g is continuous on the unit square except on the line $y = x$. However, this line has area zero. By a corollary to the Lebesgue criterion for Riemann integrability, a bounded function that is continuous almost everywhere on a closed, bounded set is Riemann integrable over that set.

Let $A = \{(x, y) \in [0, 1]^2 || x - y| < \varepsilon\}$, then area of A is less than 2ε , write $B = \{(x, y) \in [0, 1]^2 || x - y| \geq \varepsilon\}$,we know g continue on $B, \forall \delta > 0$ we can find partition such that difference between upper Darboux sum of B and lower Darboux sum of B less than δ.

On the other hand $-1 \leq g(x, y) \leq 1$,difference between upper Darboux sum of A and lower Darboux sum of A less than 4ε.

Thus , we know difference between upper Darboux sum of $A \cup B$ and lower Darboux sum of $A \cup B$ less than $4\varepsilon + \delta$, $[0, 1]^2 = A \cup B$, which prove Riemann integrable .

□

6. (a) Let C be the curve defined by the equation

$$(2x + 5y - 1)^2 + (3x - 7y + 8)^2 = 1.$$

i. (5 points) What is the shape of C?

Solution. The given equation is of the form

$$(2x + 5y - 1)^2 + (3x - 7y + 8)^2 = 1,$$

which represents an ellipse. To confirm this, we let

$$\begin{cases} u = 2x + 5y \\ v = 3x - 7y, \end{cases}$$

and inear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ sent $(1, 0) \to (2, 5); (0, 1) \to (3, -7)$,

We find A sent ellipse(circle) $C' : (u - 1)^2 + (v + 8)^2 = 1$ to $C : (2x + 5y - 1)^2 + (3x - 7y + 8)^2 = 1$.

Which means C is ellipse .

□

ii. (10 points) Find the area of the region enclosed by C.

Solution. The area of an ellipse is given by the formula:

$$\text{Area} = \pi \cdot a \cdot b,$$

where a and b are the semi-major and semi-minor axes.

on the other Hand

$$\int_C dx\, dy = \int_{C'} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv$$

$$\int_{C'} du\, dv = \pi 1^2 = \pi$$

difference between area of C and C' is Jacobian (determinant) $\frac{\partial(x,y)}{\partial(u,v)}$.

$$|\frac{\partial(x, y)}{\partial(u, v)}| = |detA|^{-1} = \frac{1}{29}.$$

Hence, the area of the region enclosed by C is:

$$\boxed{\frac{\pi}{29}}.$$

□