Recall :

THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the triple integral of the divergence $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{Divergence integral}}. \qquad (2)$$

is the 3-dim version of

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy}_{\text{Divergence integral}} \qquad (4)$$

Q : What is the 3-dim version of

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of (curl $\mathbf{F}$) $\cdot\, \mathbf{k}$ over R.

?

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C M\, dx + N\, dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy}_{\text{Curl integral}} \qquad (3)$$

Note $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \nabla \times \vec{F} \cdot (0,0,1)$ for $R \subset xy$ plane,

A :

THEOREM 5 Stokes' Theorem

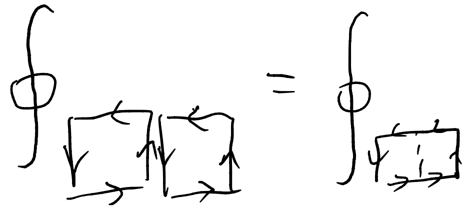
The circulation of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S.

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}} \qquad (4)$$

Informal Proof: Divide S into small pieces $\{R_i\}$

Note $\oint_C \vec{F}\,d\vec{r} = \sum_i \oint_{\partial R_i} \vec{F}\,dr$

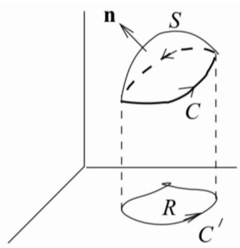
because



If we let all the rectangles be small enough, then every R_i can be approximated by a plane region.

It suffices to show stokes's thm for every R_i. Actually it's exactly Green thm if R_i is a plane region.

outer integral $= -\frac{}{2}\cos\theta\Big]_0 = -\pi$, which checks.



Example 2. Suppose $\mathbf{F} = x^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and S is given as the graph of some function $z = g(x, y)$, oriented so $\mathbf{n}$ points upwards.

Show that $\oint_C \mathbf{F}\cdot d\mathbf{r}$ = *area of R*, where C is the boundary of S, compatibly oriented, and R is the projection of S onto the xy-plane.

Solution. We have curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & x & z^2 \end{vmatrix} = \mathbf{k}$. By Stokes' theorem, (cf. V9, (12))

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R \mathbf{n}\cdot\mathbf{k}\,\frac{dA}{|\mathbf{n}\cdot\mathbf{k}|},$$

since $\mathbf{n}\cdot\mathbf{k} > 0$, $|\mathbf{n}\cdot\mathbf{k}| = \mathbf{n}\cdot\mathbf{k}$; therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_R dA = \text{area of } R\,.$$

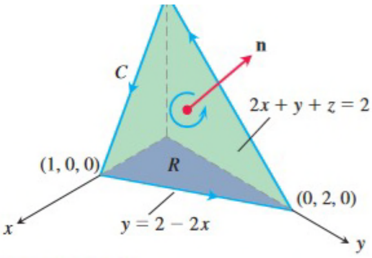


FIGURE 16.64 The planar surface in Example 9.

EXAMPLE 9 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F} = xz\mathbf{i} + xy\mathbf{j} + 3xz\mathbf{k}$ and C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant, traversed counterclockwise as viewed from above (Figure 16.64).

Solution The plane is the level surface $f(x, y, z) = 2$ of the function $f(x, y, z) = 2x + y + z$. The unit normal vector

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2\mathbf{i} + \mathbf{j} + \mathbf{k})}{|2\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{6}}\left(2\mathbf{i} + \mathbf{j} + \mathbf{k}\right)$$

is consistent with the counterclockwise motion around C. To apply Stokes' Theorem, we find

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\mathbf{j} + y\mathbf{k}.$$

On the plane, z equals $2 - 2x - y$, so

$$\nabla \times \mathbf{F} = (x - 3(2 - 2x - y))\mathbf{j} + y\mathbf{k} = (7x + 3y - 6)\mathbf{j} + y\mathbf{k}$$

and

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}}\left(7x + 3y - 6 + y\right) = \frac{1}{\sqrt{6}}\left(7x + 4y - 6\right).$$

The surface area element is

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\,dA = \frac{\sqrt{6}}{1}\,dx\,dy.$$

The circulation is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \qquad \text{Stokes' Theorem, Eq. (4)}$$

$$= \int_0^1 \int_0^{2-2x} \frac{1}{\sqrt{6}}\left(7x + 4y - 6\right)\sqrt{6}\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} (7x + 4y - 6)\, dy\, dx = -1. \qquad \blacksquare$$

Remark: To evaluate $\oint_C \vec{F} \cdot d\vec{r}$, if C is "bad", but C bounds something "good", we may use Stokes' thm. It doesn't depend on the choice of "something".