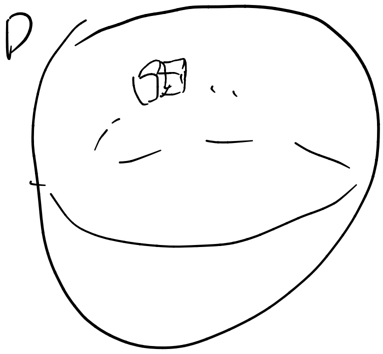
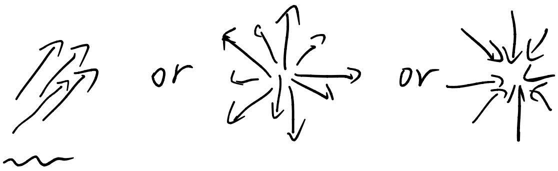
THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the triple integral of the divergence $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{Divergence integral}}. \tag{2}$$

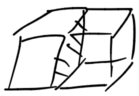
Informal Proof: Cut D into many pieces. If a piece is extremely small, it can be approximated by a small ball. Meanwhile $\vec{F}$ on this piece can be approximated by

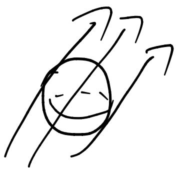




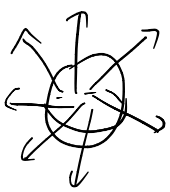
constant or $cx\vec{i}+cy\vec{j}+cz\vec{k}$ or $-cx\vec{i}-cy\vec{j}-cz\vec{k}$

If the divergence thm hold for all the pieces, they by gluing them back, it also holds for entire D, since the integrals cancelled out on the common surface.





: The flux is clearly 0. Also, $\nabla \cdot F = 0$.



: Suppose the sphere has radius r. Then $\vec{n} = \dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{r}$

$$\vec{F} \cdot \vec{n} = c \cdot \frac{x^2+y^2+z^2}{r} = cr.$$

$$\oiint_{\partial D} \vec{F} \cdot \vec{n}\, d\sigma = \oiint_{\partial D} cr\, d\sigma = 4c\pi r^3.$$

On the other hand, $\nabla \cdot \vec{F} = c\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) \equiv 3c$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 3c\, dV = 3c \cdot \frac{4}{3}\pi r^3 = 4c\pi r^3$$

Ex

Let's say we wanted to evaluate the flux of the following vector field defined by $\mathbf{F} = 2x^2\mathbf{i} + 2y^2\mathbf{j} + 2z^2\mathbf{k}$ bounded by the following inequalities:

$$\{0 \le x \le 3\}, \{-2 \le y \le 2\}, \{0 \le z \le 2\pi\}$$

By the divergence theorem,

$$\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, \mathrm{d}S.$$

We now need to determine the divergence of $\mathbf{F}$. If $\mathbf{F}$ is a three-dimensional vector field, then the divergence of $\mathbf{F}$ is given by $\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \mathbf{F}$.

Thus, we can set up the following flux integral $I = \oiint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$, as follows:

$$\begin{aligned} I &= \iiint_V \nabla \cdot \mathbf{F}\, \mathrm{d}V \\ &= \iiint_V \left(\frac{\partial \mathbf{F_x}}{\partial x} + \frac{\partial \mathbf{F_y}}{\partial y} + \frac{\partial \mathbf{F_z}}{\partial z}\right) \mathrm{d}V \\ &= \iiint_V (4x + 4y + 4z)\, \mathrm{d}V \\ &= \int_0^3 \int_{-2}^2 \int_0^{2\pi} (4x + 4y + 4z)\, \mathrm{d}V \end{aligned}$$

Now that we have set up the integral, we can evaluate it.

$$\int_0^3 \int_{-2}^2 \int_0^{2\pi} (4x + 4y + 4z)\, \mathrm{d}V = \int_{-2}^2 \int_0^{2\pi} (12y + 12z + 18)\, \mathrm{d}y\, \mathrm{d}z$$

$$= \int_0^{2\pi} 24(2z+3)\, dz$$

$$= 48\pi(2\pi+3)$$

Divergence thm in arbitrary dimensions:

$$\overbrace{\int\cdots\int_U}^{n} \nabla\cdot\vec{F}\, dV = \oint\cdots\oint_{\partial U} \vec{F}\cdot\vec{n}\, ds$$

$n=1$: $\vec{F}$ is a $\mathbb{R}$-valued function, say f. $\nabla\cdot\vec{F} = \frac{\partial f}{\partial t}$

$$\int_{[a,b]} \frac{\partial f}{\partial t}\, dt = f(b) - f(a)$$

the fundamental thm of calculus.

$n=2$ $$\iint_R \nabla\cdot\vec{F}\, d\sigma = \oint_{\partial R} \vec{F}\cdot\vec{n}\, ds$$

Green thm.

In the future (Generalized Stokes's thm):

$$\int_{\partial M} \omega = \int_M d\omega$$

manifold

differential form