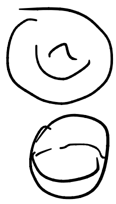
Q: What is a curve in $\mathbb{R}^n$

A: A continuous map $\gamma : [0,1] \longrightarrow \mathbb{R}^n$?
$(0,1]$
$[0,1)$
$(0,1)$

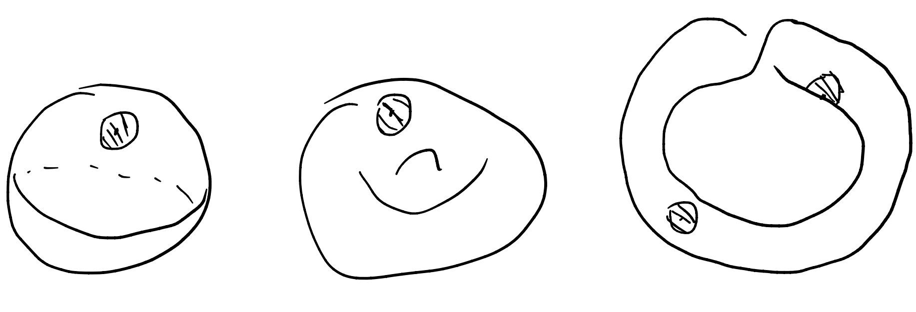
Q: What is a surface in $\mathbb{R}^n$?

Guess: A continuous map $S : ? \longrightarrow \mathbb{R}$

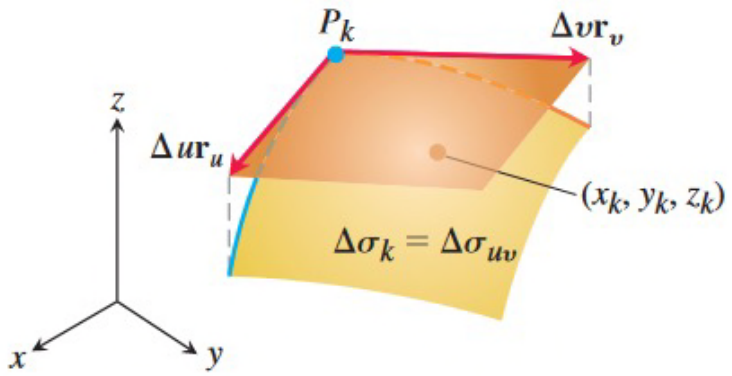
By analogy, ? should be . However,  cannot be obtained in this way.

A: A surface in $\mathbb{R}^n$ is a subset S s.t. $\forall x \in S$, $\exists r > 0$ s.t. $B_r(x) \cap S$ is the "same" as  or 

X and Y being the "same" means $\exists$ a bijection $f : X \to Y$ s.t. f and f^{-1} are continuous. Formally, f is called a <u>homeomorphism</u> and X, Y are <u>homeomorphic</u> to each other

Ex 

Surfaces in $\mathbb{R}^3$



$\Longrightarrow \Delta\sigma_{uv} \approx |\vec{r}_u \times \vec{r}_v|\, du\, dv$

Take partitions of S. Def $\iint_S G(x,y,z)\,d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\,d\sigma_k$.

Sometimes it can be very hard to describe a surface explicitly.

If a surface in $\mathbb{R}^3$ can be described by one of the following ways, the we have the formulas:

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\,d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv. \qquad (2)$$

Line integral:

$\oint_C G(x,y,z)\,ds$

$= \int_a^b G(f(t), g(t), h(t))\,|\vec{v}(t)|\,dt$

2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

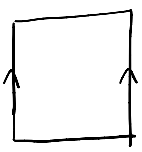
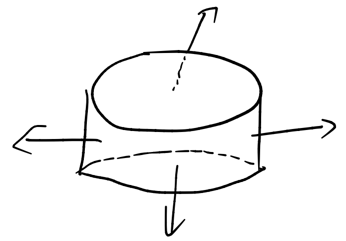
$$\iint_S G(x, y, z)\,d\sigma = \iint_R G(x, y, z)\,\frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\,dA, \qquad (3)$$

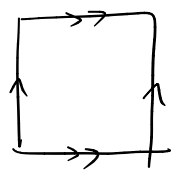
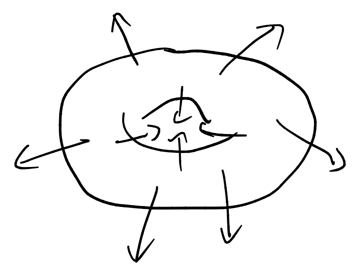
where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

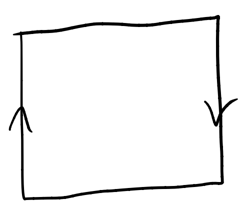
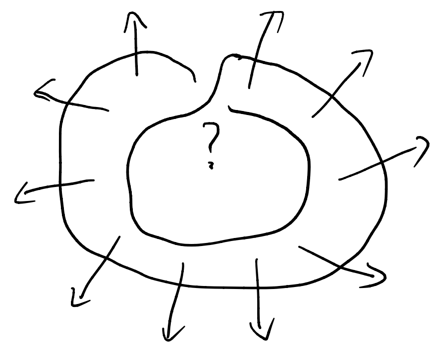
3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

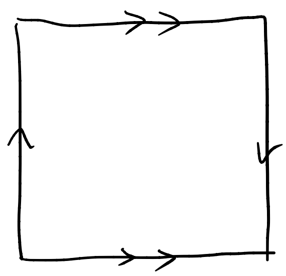
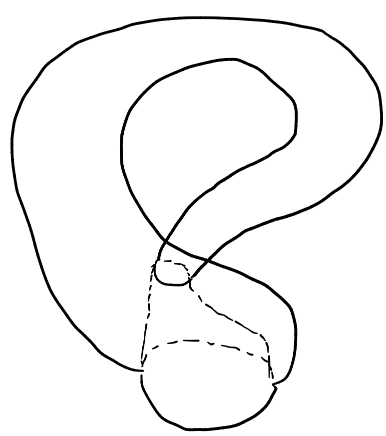
$$\iint_S G(x, y, z)\,d\sigma = \iint_R G(x, y, f(x, y))\,\sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dx\,dy. \qquad (4)$$

Def Let S be a surface in $\mathbb{R}^3$. At every pt of S, there are two choices of unit normal vectors. If $\exists$ a continuous choice of unit normal vectors at all the pts of S, then S is called <u>orientable</u>

Ex 1.  

2.  

non-Ex 1.  

2.  

Using the definition above, we cannot say whether Klein bottle is orientable or not because it cannot be embeded in $\mathbb{R}^3$. In fact, we can use more general definitions to say it is not orientable.