Pullback of differential forms:

$$g: U \subset \mathbb{R}^m \to \mathbb{R}^n$$

$$\omega = \sum_I f_I \, dx_I \in A^k(\mathbb{R}^n)$$

$$g^*(\omega) = \sum_I (f_I \circ g) \, dg_I \in A^k(\mathbb{R}^m)$$

Remark: 1. We think functions on $\mathbb{R}^n$ are 0-forms. In this case $g^*(f) = f \circ g \in A^0(\mathbb{R}^m) \ \forall f \in A^0(\mathbb{R}^n)$.

2. We think x_i is a function: $\mathbb{R}^n \to \mathbb{R}$. For example, let $p = (a,b,c) \in \mathbb{R}^3$. Then $x_1(p) = a$, $x_2(p) = b$, $x_3(p) = c$. Thus $x_i \in A^0(\mathbb{R}^n)$ and $dx_i \in A^1(\mathbb{R}^n)$.

3. Computing integrals in different coordinates is equivalent to pullback of differential forms

Ex 1. $g: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad \omega = xy\,dx + 2z\,dy - y\,dz \in \Omega^k(\mathbb{R}^3)$

$(u,v) \mapsto (uv, u^2, 3u+v)$

$$g^*(\omega) = uv \cdot u^2 \, d(uv) + 2(3u+v)\, d(u^2) - u^2 \, d(3u+v)$$

$$= u^3 v (u\,dv + v\,du) + (6u+2v) \cdot 2u\,du - u^2(3\,du + dv)$$

$$= (u^3v^2 + 9u^2 + 4uv)\,du + (u^4 v - u^2)\,dv$$

2. $g: \mathbb{R}^n \to \mathbb{R}^n$ differentiable. $\omega = dy_1 \wedge \cdots \wedge dy_n$

$$g^*(\omega) = \sum_{j_1 \cdots j_n} \frac{\partial g_1}{\partial x_{j_1}} \cdots \frac{\partial g_n}{\partial x_{j_n}} \, dx_{j_1} \wedge \cdots \wedge dx_{j_n}$$

Note $dx_{j_1}\wedge\cdots\wedge dx_{j_n} \neq 0$ iff $j_1,\cdots,j_n$ are distinct. so

$$
\begin{aligned}
☺ &= \sum_{\sigma\in S_n} \frac{\partial g_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial g_n}{\partial x_{\sigma(n)}}\, dx_{\sigma(1)}\wedge\cdots\wedge dx_{\sigma(n)} \\
&= \sum_{\sigma\in S_n} \frac{\partial g_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial g_n}{\partial x_{\sigma(n)}}\, \mathrm{sign}(\sigma)\, dx_1\wedge\cdots\wedge dx_n \\
&= (\det Dg)\, dx_1\wedge\cdots\wedge dx_n
\end{aligned}
$$

This implies the formula of change of varibles:

$$\int_{g(U)} f\, dy_1\cdots dy_n = \int_U f\circ g\cdot|\det Dg|\, dx_1\cdots dx_n$$

3. $\vec{F}:\mathbb{R}^3\to\mathbb{R}^3$, $\vec{F}=(F_1, F_2, F_3)$

$\eta = F_1\, dy\wedge dz - F_2\, dx\wedge dz + F_3\, dx\wedge dy$

Then
$$
\begin{aligned}
d\eta &= dF_1\wedge dy\wedge dz - dF_2\wedge dx\wedge dz + dF_3\wedge dx\wedge dy \\
&= \frac{\partial F_1}{\partial x}\, dx\wedge dy\wedge dz - \frac{\partial F_2}{\partial y}\, dy\wedge dx\wedge dz + \frac{\partial F_3}{\partial z}\, dz\wedge dx\wedge dy \\
&= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\wedge dy\wedge dz \\
&= \mathrm{div}\,\vec{F}\, dx\wedge dy\wedge dz
\end{aligned}
$$

Recall (Generalized Stokes' thm): $\int_{\partial M} \omega = \int_M d\omega$

Let $\omega=\eta$, We have $\int_{\partial M} F_1\, dy\wedge dz - F_2\, dx\wedge dz + F_3\, dx\wedge dy = \int_M \mathrm{div}\,\vec{F}\, dx\wedge dy\wedge dz$

This is the Divergence Thm.

Similarly, by letting $\omega = F_1 dx + F_2 dy + F_3 dz$, we have

$$d\omega = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy.$$

This implies Stokes' Thm.