MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Problem Set 6
due on Dec 2, 2024 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: All curves, surfaces and vector fields are inside $\mathbb{R}^3$. We will use U to denote an open subset of $\mathbb{R}^3$.

Problems to hand in

1. Compute the flux $\int_S (\nabla \times F) \cdot \vec{n}\, d\sigma$ where
   (a) $F(x, y, z) = (x^2 + y, yz, x - z^2)$ and S is the triangle defined by the plane $2x + y + 2z = 2$ inside the first octant, oriented by the unit normal pointing away from the origin.
   (b) $F(x, y, z) = (x, y, 0)$ and S is the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$, oriented by the upward pointing normal.
2. Let $F(x, y, z) = (ye^z, xe^z, xye^z)$ and C be a simple closed curve which is the boundary of a surface S. Show that $\int_C F \cdot\ d\vec{r} = 0$.
3. Find $\iint_S F \cdot \vec{n}\, d\sigma$ where
   (a) $F(x, y, z) = (2x, y^2, z^2)$ and S is the unit sphere centered at the origin, oriented by the outward unit normal;
   (b) $F(x, y, z) = (x + y, y + z, x + z)$ and S is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, oriented by the outward unit normal.
4. Given a simple closed curve C that bounds a region D in $\mathbb{R}^2$ and a smooth vector field $\vec{F} = (P, Q)$, the *Flux* of $\vec{F}$ across C is defined as $\oint_C \vec{F} \cdot \hat{n} ds := \oint_C -Qdx + Pdy$. Deduce the following 2-dimensional version of divergence theorem from Green's theorem:
$$\oint_C \vec{F} \cdot \hat{n} ds = \iint_D \nabla \cdot \vec{F}\, dA$$
5. Let $\omega = y^2\, dy \wedge dz + x^2\, dz \wedge dx + z^2\, dx \wedge dy$, and M be the solid paraboloid $0 \le z \le 1 - x^2 - y^2$. Evaluate $\int_{\partial M} \omega$ directly and by applying Stokes' Theorem.
6. Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4\ :\ x_1^2 + x_2^2 + x_3^2 \le x_4 \le 1\}$, with the standard orientation inherited from $\mathbb{R}^4$. Evaluate
$$\int_{\partial M} (x_1^3 x_2^4 + x_4)\, dx_1 \wedge dx_2 \wedge dx_3.$$

Suggested Exercises

1. A function $f : U \to \mathbb{R}$ is said to be harmonic if $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

(a) Prove that the functions $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ and $f(x,y,z) = x^2 - y^2 + 2z$ are harmonic on their maximal domain of definition.

(b) Show that $\nabla \cdot (\nabla f) = 0$ if f is harmonic.

2. Prove that $F(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ satisfies $\nabla \cdot F = 0$ and $\nabla \times F = 0$ on $\mathbb{R}^3 \setminus \{0\}$.

3. Prove the following identities:

   (a) $\nabla \times (\nabla f) = 0$ for any C^2 function $f : U \to \mathbb{R}$;

   (b) $\nabla \cdot (\nabla \times F) = 0$ for any C^2 vector field $F : U \to \mathbb{R}^3$.

   (c) $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$ for any vector fields F, G.

   (d) $\nabla \cdot (\nabla f \times \nabla g) = 0$ for any functions f, g.

4. Verify Stokes theorem for

   (a) $F(x,y,z) = (z,x,y)$ and S defined by $z = 4 - x^2 - y^2$ and $z \geq 0$;

   (b) $F(x,y,z) = (x,z,-y)$ and S is the portion of the sphere of radius 2 centered at the origin with $y \geq 0$;

   (c) $F(x,y,z) = (y+x, x+z, z^2)$ and S is the portion of the cone $z^2 = x^2 + y^2$ with $0 \leq z \leq 1$.

5. Let C be a closed curve which is the boundary of a surface S. Prove that

   (a) $\int_C f\nabla g \cdot \, d\vec{r} = \iint_S (\nabla f \times \nabla g) \cdot \vec{n} \, d\sigma$;

   (b) $\int_C (f\nabla g + g\nabla f) \cdot \, d\vec{r} = 0$.

6. Find $\iint_S F \cdot \vec{n} \, d\sigma$ where

   (a) $F(x,y,z) = (x^3, y^3, z^3)$ and S is the unit sphere centered at the origin, oriented by the outward unit normal;

   (b) $F(x,y,z) = (x+y, y+z, x+z)$ and S is the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$, oriented by the upward unit normal;

   (c) $F(x,y,z) = (2x, 3y, z)$ and S is the closed surface consisting of the cylinder $x^2 + y^2 = 4$ and the planes $z = 1$, $z = 3$, oriented by the outward unit normal;

7. Suppose Ω is the interior of a closed surface S. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^2 functions. Prove the following *Green's identities*:

   (a) $\iint_S (f\nabla g) \cdot \vec{n} \, d\sigma = \iiint_\Omega (f\Delta g + \nabla f \cdot \nabla g) \, dV$;

   (b) $\iint_S (f\nabla g - g\nabla f) \cdot \vec{v} \, d\sigma = \iiint_\Omega (f\Delta g - g\Delta f) \, dV$;

   Here, $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

8. Let $\Omega \subset \mathbb{R}^3$ be a bounded open subset with boundary $\partial\Omega = S$ which is a closed surface, oriented by the outward unit normal $\vec{n}$. Let $F(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$. Assume that $0 \notin S$.

   (a) Suppose that $0 \notin \Omega$. Show that

$$\iint_S F \cdot \vec{n} \, d\sigma = 0.$$

(a) Suppose that $0 \in \Omega$. Show that

$$\iint_S F \cdot \vec{n} \, d\sigma = 4\pi.$$

9. Can there be a function f so that df is the given 1-form ω (everywhere ω is defined)? If so, find f.

   (a) $\omega = y\, dx + z\, dy + x\, dz$

   (b) $\omega = (x^2 + yz)\, dx + (xz + \cos y)\, dy + (z + xy)\, dz$

   (c) $\omega = \frac{x}{x^2+y^2}\, dx + \frac{y}{x^2+y^2}\, dy$

   (d) $\omega = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$

10. For each of the following k-forms ω, can there be a $(k-1)$-form η (defined wherever ω is) so that $d\eta = \omega$?

    (a) $\omega = z\, dx \wedge dy$

    (b) $\omega = z\, dx \wedge dy + y\, dx \wedge dz + z\, dy \wedge dz$

    (c) $\omega = x\, dx \wedge dy + y\, dx \wedge dz + z\, dy \wedge dz$

    (d) $\omega = (x^2 + y^2 + z^2)^{-1}(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$

11. In each of the following, compute the pullback $g^*\omega$ and verify that $g^*(d\omega) = d(g^*\omega)$:

    (a) $g(v) = (3\cos 2v, 3\sin 2v)$, $\omega = -y\, dx + x\, dy$

    (b) $g(u, v) = (\cos u, \sin u, v)$, $\omega = z\, dx + x\, dy + y\, dz$

    (c) $g(u, v) = (\cos u, \sin v, \sin u, \cos v)$, $\omega = (-x_3\, dx_1 + x_1\, dx_3) \wedge (-x_2\, dx_4 + x_4\, dx_2)$

Challenging Exercises

1. Let $F : U \to \mathbb{R}^3$ be a C^1 vector field defined on an open subset $U \subset \mathbb{R}^3$. Fix $p \in U$. Denote $B_r(p)$ be the closed ball of radius $r > 0$ centered at p and $S_r(p) = \partial B_r(p)$ be the sphere of radius $r > 0$ centered at p, with outward pointing unit normal $\vec{n}$. Prove that

$$(\nabla \cdot F)(p) = \lim_{r \to 0} \frac{1}{\mathrm{Vol}(B_r(p))} \iint_{S_r(p)} F \cdot \vec{n} \, d\sigma.$$

2. Let $S \subset \mathbb{R}^3$ be a surface and $F : U \to \mathbb{R}^3$ be a C^1 vector field defined on an open set $U \subset \mathbb{R}^3$ containing S. Fix $p \in S$. Denote $D_r(p) := \{x \in S \mid |x - p| \leq r\}$ and $C_r(p) = \{x \in S \mid |x - p| = r\}$. Suppose S is oriented by the unit normal $\vec{n}$ and so is $C_r(p)$ as the boundary of $D_r(p)$ (which you can assume to be C^1). Prove that

$$(\nabla \times F)(p) \cdot \vec{n}(p) = \lim_{r \to 0} \frac{1}{\mathrm{Area}(D_r(p))} \int_{C_r(p)} F \cdot d\vec{r}.$$