MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Problem Set 3
due on Oct 18, 2024 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use (r, θ), (r, θ, z) and (ρ, ϕ, θ) to denote the polar, cylindrical and spherical coordinates respectively.

Problems to hand in

1. Find the volume of the region lying above the plane $z = a$ and inside the sphere $x^2+y^2+z^2 = 4a^2$ by integrating in cylindrical coordinates and spherical coordinates.

2. (a) Find the volume of a right circular cone of base radius a and height h by integrating in cylindrical coordinates and spherical coordinates.
   (b) How about the volume of an oblique cones where the vertex also lie at height h but not necessarily directly over the center of the circular base?
   (c) In general, what is the volume of a generalized cone with a given base area A and height h?

3. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2 + y^2 = 1$, $y^2 + z^2 = 1$, and $x^2 + z^2 = 1$.

4. Let $\Omega \subset \mathbb{R}^2$ be the open subset bounded by the curve $x^2 - xy + 2y^2 = 1$. Express the integral $\int_\Omega xy\, dA$ as an integral over the unit disk in $\mathbb{R}^2$ centered at the origin.

5. Let $\Omega \subset \mathbb{R}^2$ be the open subset in the first quadrant bounded by $y = 0$, $y = x$, $xy = 1$ and $x^2 - y^2 = 1$. Evaluate the integral $\int_\Omega (x^2 + y^2)\ dA$ using the change of variables $u = xy$, $v = x^2 - y^2$.

6. Let $B^n(r)$ denote the closed ball of radius a in $\mathbb{R}^n$ centered at the origin.
   (a) Show that $\text{Vol}(B^n(r)) = \lambda_n r^n$ for some positive constant λ_n.
   (b) Compute λ_1 and λ_2.
   (c) Compute λ_n in terms of λ_{n-2}.
   (d) Deduce a formula for λ_n for general n. *(Hint: consider two cases, according to whether n is even or odd.)*

Suggested Exercises

1. Let $\Omega \subset \mathbb{R}^2$ be the region bounded below by $y = 1$ and above by $x^2 + y^2 = 4$. Evaluate

$$\int_\Omega (x^2 + y^2)^{-3/2}\ dA.$$

2. Find the area enclosed by the cardioid in $\mathbb{R}^2$ expressed in polar coordinates as $r = 1 + \cos\theta$.

3. Let $\Omega \subset \mathbb{R}^3$ be the region bounded below by the sphere $x^2+y^2+z^2 = 2z$ and above by the sphere $x^2 + y^2 + z^2 = 1$. Evaluate the integral

$$\int_\Omega \frac{z}{(x^2+y^2+z^2)^{3/2}}\, dV.$$

4. Let $\Omega \subset \mathbb{R}^2$ be the open subset lying in the first quadrant and bounded by the hyperbolas $xy = 1$, $xy = 2$ and the lines $y = x$, $y = 4x$. Evaluate the integral $\int_\Omega x^2y^3\, dA$.

5. Let $\Omega \subset \mathbb{R}^3$ be the open tetrahedron with vertices $(0,0,0)$, $(1,2,3)$, $(0,1,2)$ and $(-1,1,1)$. Evaluate the integral $\int_\Omega (x + 2y - z)\, dV$.

6. Let $\Omega \subset \mathbb{R}^2$ be the open subset bounded by $x = 0$, $y = 0$ and $x + y = 1$. Evaluate the integral $\int_\Omega \cos\left(\frac{x-y}{x+y}\right) dA$. *(Hint: note that the integrand is un-defined at the origin.)*

7. Find the volume of the solid region $\Omega \subset \mathbb{R}^3$ bounded below by the surface $z = x^2+2y^2$ and above by the plane $z = 2x + 6y + 1$ by expressing it as an integral over the unit disk in $\mathbb{R}^2$ centered at the origin.

8. Let $\Omega \subset \mathbb{R}^2$ be the open triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Evaluate the integral $\int_\Omega e^{(x-y)/(x+y)}\, dA$
   (a) using polar coordinates;
   (b) using the change of variables $u = x - y$, $v = x + y$.

Challenging Exercises

1. (a) Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of one of the following types:

   (i) $\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases}$

   (ii) $\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$

   (iii) $\begin{cases} g(e_k) = e_k & k \neq i, j \\ g(e_i) = e_j \\ g(e_j) = e_i \end{cases}$

   If U is a rectangle, show that the volume of $g(U)$ is $|\det g| \cdot vol(U)$.

   (b) Prove that $|\det g| \cdot vol(U)$ is the volume of $g(U)$ for any linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$. (*Hint: If* $\det g \neq 0$, *then* g *is the composition of linear transformations of the type considered in (a).*)

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with measure zero $\partial\Omega$. Show that for any $\epsilon > 0$, there exists a compact subset $K \subset \Omega$ such that ∂K has measure zero and $\text{Vol}(\Omega \setminus K) < \epsilon$.