§ Differential Forms.

- Basic 1-form in $\mathbb{R}^n$: $dx_1, \dots, dx_n$ "formal symbol".

  General 1-form: $\boxed{\omega = f_1\,dx_1 + \cdots + f_n\,dx_n}$, f_i smooth functions

  "wedge"

- Basic 2-form: $dx_i \wedge dx_j$ $\quad 1 \le i < j \le n$. Total # $= \binom{n}{2} = \frac{n(n-1)}{2}$.

  General 2-form: $\boxed{\omega = \sum_{i<j} f_{ij}\, dx_i \wedge dx_j}$, $f_{i,j}$ smooth functions.

In general, Basic k-form: $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. $\forall\, 1 \le i_1 < \cdots < i_k \le n$. Total # $= \binom{n}{k}$.

General k-form: $\boxed{\omega = \sum_I \omega_I\, dx_I = \sum_I f_{i_1, \dots, i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}}$. $I = \{i_1 < \cdots < i_k\}$.

Denote $\Omega^k(\mathbb{R}^n) = \{(\text{general})\ k\text{-forms}\}$. "Vector space with $C^\infty(\mathbb{R}^n)$-coeff"

with basis $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. $\dim = \binom{n}{k}$

Operations on differential forms.

- Addition : k-form + k-form = k-form

$$\sum_I f_I\, dx_I + \sum_I g_I\, dx_I = \sum_I (f_I + g_I)\, dx_I$$

- Wedge Product : Let $\boxed{dx_i \wedge dx_j = -dx_j \wedge dx_i}$ anti-commutative

($\Rightarrow dx_i \wedge dx_i = 0$)

Satisfies : $\boxed{(dx_i \wedge dx_j) \wedge dx_k = dx_i \wedge (dx_j \wedge dx_k)}$ associative

This extends to a bilinear map : $\Omega^k(\mathbb{R}^n) \wedge \Omega^l(\mathbb{R}^n) \longrightarrow \Omega^{k+l}(\mathbb{R}^n)$

Ex: $dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz = dy \wedge dz \wedge dx = -dz \wedge dy \wedge dx = dz \wedge dx \wedge dy = -dx \wedge dz \wedge dy$

Ex: $\omega = P\, dy \wedge dz - Q\, dx \wedge dz + R\, dx \wedge dy \in \Omega^2(\mathbb{R}^3)$, $\eta = L\, dx + M\, dy + N\, dz \in \Omega^1(\mathbb{R}^3)$

$$\omega \wedge \eta = \underbrace{(PL + QM + RN)}_{\text{"dot product"}}\, dx \wedge dy \wedge dz = \eta \wedge \omega$$

Prop: $\omega \in \Omega^k(\mathbb{R}^n)$, $\eta \in \Omega^l(\mathbb{R}^n)$, then $\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega$

pf: By linearity, enough to prove the case $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\eta = g\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$

$\omega \wedge \eta = f \cdot g\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \cdots \wedge dx_{j_l}$, $\eta \wedge \omega = g f\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

total $k \cdot l$ transpositions $\rightsquigarrow (-1)^{kl}$. $\square$

- **Exterior derivative**: Let · $\boxed{df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n}$.

  · $d(dx_i) = 0$

This extends to a linear map $d : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$.

Ex: $\omega = P\, dy \wedge dz - Q\, dx \wedge dz + R\, dx \wedge dy \in \Omega^2(\mathbb{R}^3)$.

$$d\omega = dP \wedge dy \wedge dz - dQ \wedge dx \wedge dz + dR \wedge dx \wedge dy$$

$$= \underbrace{(P_x + Q_y + R_z)}_{\text{"divergence"}}\, dx \wedge dy \wedge dz.$$

Prop: (Leibnitz Rule): $\omega \in \Omega^k(\mathbb{R}^n)$. $\eta \in \Omega^l(\mathbb{R}^n)$

then $$\boxed{d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta}$$

Theorem: $d^2\omega = 0 \quad \forall \omega \in \Omega^k(\mathbb{R}^n)$. $d^2 = 0$.

Essentially equivalen to Mixed derivative Thm: $f_{ij} = f_{ji}$

Def: $\omega \in \Omega^k(\mathbb{R}^n)$ closed form if $d\omega = 0$.

exact form if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(\mathbb{R}^n)$

exact form $\Rightarrow$ closed form $\left(d\omega = d(d\eta) = 0\right)$.

$\not\Leftarrow$

In $\mathbb{R}^3$: **Differential Forms** $\quad\quad$ **Vector Calculus**

f $\quad$ 0-form Ω^0 $\xleftrightarrow{=}$ Scalar function $\quad f$

$$df = f_x\,dx + f_y\,dy + f_z\,dz \quad \Big\downarrow d \qquad\qquad \text{grad}\Big\downarrow \qquad \nabla f: \ \langle f_x, f_y, f_z\rangle = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

1-form Ω^1 $\xleftrightarrow{dx \leftrightarrow \hat{i},\ dy \leftrightarrow \hat{j},\ dz \leftrightarrow \hat{k}}$ Vector field

$P\,dx + Q\,dy + R\,dz$ $\qquad\qquad$ $\vec{F} = \langle P, Q, R\rangle$

$$(R_y - Q_z)\,dy\wedge dz - (P_z - R_x)\,dx\wedge dz + (Q_x - P_y)\,dx\wedge dy \quad \Big\downarrow d \qquad \text{curl}\Big\downarrow \qquad \nabla\times\vec{F}: \ \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$$

2-form Ω^2 $\xleftrightarrow{dy\wedge dz \leftrightarrow \hat{i},\ -dx\wedge dz \leftrightarrow \hat{j},\ dx\wedge dy \leftrightarrow \hat{k}}$ Vector field

$P\,dy\wedge dz - Q\,dx\wedge dz + R\,dx\wedge dy$ $\qquad\qquad$ $\vec{F} = \langle P, Q, R\rangle$

$$\Big\downarrow d \qquad \text{div}\Big\downarrow \qquad \nabla\cdot\vec{F}$$

$(P_x + Q_y + R_z)\,dx\wedge dy\wedge dz$ $\quad$ 3-form Ω^3 $\xleftrightarrow{dx\wedge dy\wedge dz \ \leftrightarrow\ 1}$ Scalar function $\quad P_x + Q_y + R_z$

Rmk: $\quad d^2 = 0 \iff \nabla\times(\nabla f) = 0,\ \ \nabla\cdot(\nabla\times\vec{F}) = 0$

- **Pull-back**: $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$. Smooth map $\varphi: U \longrightarrow V$

$(x_1, \cdots, x_m) \rightsquigarrow (y_1 = \varphi_1(x_1, \cdots, x_m), \cdots y_n = \varphi_n(x_1, \cdots, x_m))$

Given $\omega \in \Omega^k(V)$, define $\varphi^*\omega \in \Omega^k(U)$:

- If $\omega = f \in C^\infty(V) = \Omega^0(V)$, let $\boxed{\varphi^*\omega = f \circ \varphi} \in C^\infty(U) = \Omega^0(U)$

- If $\omega = dy_i \in \Omega^1(V)$ Let $\boxed{\varphi^*\omega = d\,\varphi_i(x_1, \cdots, x_m) = \sum_{j=1}^{m} \frac{\partial \varphi_i}{\partial x_j} dx_j} \in \Omega^1(V)$

- In general, $\omega = \sum f_{i_1, \cdots, i_k} dy_{i_1} \wedge \cdots \wedge dy_{i_k} \in \Omega^k(U)$

Let $\boxed{\varphi^*\omega = \sum \varphi^*(f_{i_1, \cdots, i_k}) \, \varphi^*(dy_{i_1}) \wedge \cdots \wedge \varphi^*(dy_{i_k})} = \sum g_{j_1, \cdots, j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k} \in \Omega^k(U)$

Ex: $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(r, \theta) \rightsquigarrow (x = r\cos\theta,\ y = r\sin\theta)$

For $\omega = dx \wedge dy$, $\varphi^*\omega = \varphi^*(dx) \wedge \varphi^*(dy) = (\cos\theta \cdot dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$

$= r\, dr \wedge d\theta$

Ex: For general $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$(x_1, \cdots, x_n) \rightsquigarrow (y_1(x_1 \cdots, x_n), \cdots, y_n(x_1 \cdots, x_n))$

For $\omega = dy_1 \wedge \cdots \wedge dy_n$, $\varphi^*\omega = \left(\sum_j \frac{\partial y_1}{\partial x_j} dx_j\right) \wedge \cdots \wedge \left(\sum_j \frac{\partial y_n}{\partial x_j} dx_j\right)$

$$= \underbrace{\det\left(\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}\right)} dx_1 \wedge \cdots \wedge dx_n$$

Jacobian matrix

Integration of differential form.

Suppose $R \subset \mathbb{R}^n$ a region w/ ordered basis $(x_1, \cdots, x_n)$. $\Omega^n(R)$: $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$

Define $$\boxed{\int_R \omega := \int_R f \underbrace{dx_1 \cdots dx_n}_{dV}}$$

Change of Variable: $\varphi: U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^n$

$(x_1, \cdots, x_n) \rightsquigarrow (y_1, \cdots, y_n)$

We learnt formula: $$\int_V f(y_1, \cdots, y_n)\, dy_1 \cdots dy_n = \int_U f \circ \varphi(x_1, \cdots, x_n) \left| \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} \right| dx_1 \cdots dx_n$$

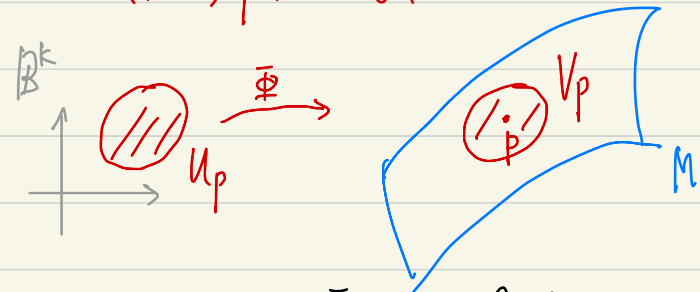
$\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

In diff. form:
$(\omega = f\, dy_1 \wedge \cdots \wedge dy_n)$ $$\boxed{\int_V \omega = \int_U \varphi^* \omega}$$

Def: M k-dim manifold in $\mathbb{R}^n$ if $\forall p \in M$, $\exists$ an open nghd $p \in V_p \subset M$

and a diffeomorphism $\underset{\sim\sim\sim\sim\sim\sim\sim}{\Phi : U_p \xrightarrow{\cong} V_p}$ for some open set $U_p \subset \mathbb{R}^k$.

$\nwarrow$ a (local) parametrization



Def: • Given $\omega \in \Omega^k(V)$

let $\boxed{\displaystyle\int_V \omega := \int_{U = \Phi^{-1}(V) \subset \mathbb{R}^k} \Phi^* \omega}$

• More generally, suppose $\bigcup_\alpha V_\alpha = M$, and parametrizations $\Phi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha$

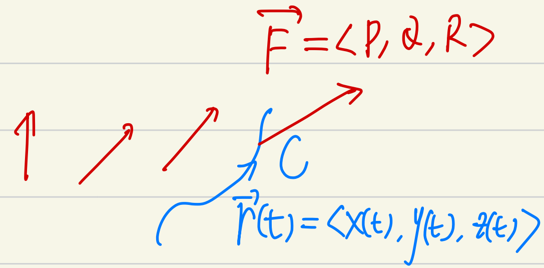
Let $\{\rho_\alpha\}$ partition of unity sub. to $\{V_\alpha\}$.

For $\omega \in \Omega^k(M)$,

$\omega = \sum_\alpha \rho_\alpha \cdot \omega \in \Omega^k(V_\alpha)$, define $\boxed{\displaystyle\int_M \omega := \sum_\alpha \int_{U_\alpha} \Phi_\alpha^* (\rho_\alpha \cdot \omega)}$

Ex: $\Phi : [a,b] \subset \mathbb{R} \longrightarrow C \subset \mathbb{R}^3$

$t \rightsquigarrow (x(t), y(t), z(t))$



For $\omega = P\,dx + Q\,dy + R\,dz$

$$\int_C \omega = \int_{[a,b]} \Phi^*(\omega) = \int_{[a,b]} P\,x'(t) + Q\,y'(t) + R\,z'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}.$$

Ex: $\Phi : U \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$

$(u,v) \rightsquigarrow (x(u,v), y(u,v), z(u,v))$

For $\omega = P\,dy\wedge dz - Q\,dx\wedge dz + R\,dx\wedge dy$

$$\Phi^*\omega = \underbrace{P\cdot \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} - Q \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + R \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}}\,du\,dv$$

On the other hand, for $\vec{F} = \langle P, Q, R \rangle$

$$\int_S \vec{F}\cdot\hat{n}\,d\sigma = \int_U \underbrace{\langle P, Q, R \rangle \cdot (\vec{r}_u \times \vec{r}_v)}\,du\,dv \quad = \quad \Rightarrow \int_S \omega = \int_U \Phi^*\omega = \int_U \text{''}$$

Generalized Stokes Thm: M k-dim mfld, possibly with boundary, $\partial M = (k-1)$-dim.

$\omega \in \Omega^{k-1}(M)$, $d\omega = \Omega^{k}(M)$.

Then $\boxed{\int_M d\omega = \int_{\partial M} \omega}$

Ex: Fund. Thm of line integral, Green's thm, Stokes' thm, divergent thm.

Corollary: For $\omega \in \Omega^k(M)$, M k-dim mfld, $\int_M \omega = 0$ if
$\omega = d\eta$ exact and $\partial M = \phi$
or $d\omega = 0$ and $M = \partial N$ for some $(k+1)$-dim mfld N.

pf: $\int_M \underset{\sim}{\omega}^{= d\eta} = \int_{\partial M} \eta = 0$, or $\int_{M = \partial N} \omega = \int_N \cancelto{0}{d\omega} = 0$ $\square$