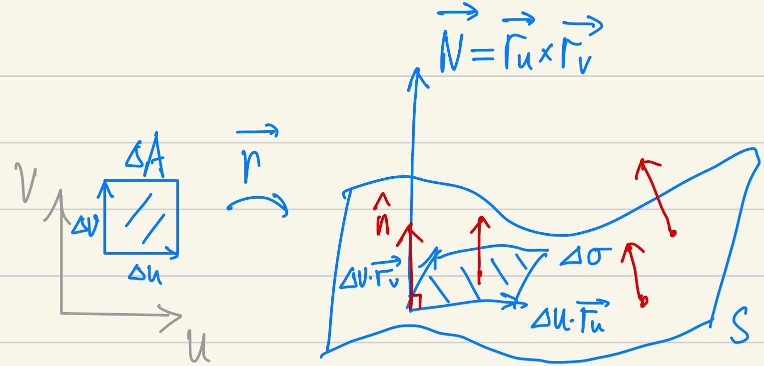
§ Surface Integral

Parametrized Surface.



$\vec r(u\ v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$

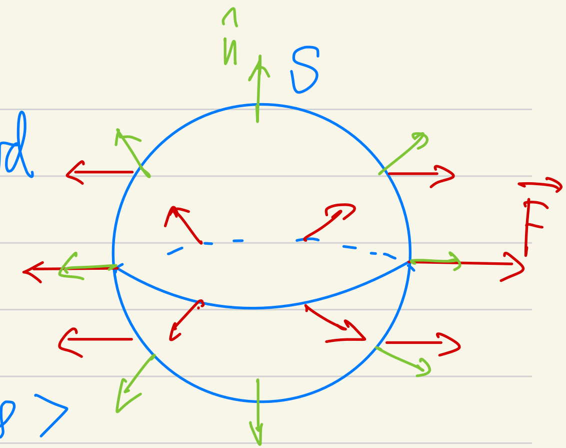
Let $\vec N = \vec r_u \times \vec r_v$: $\quad \|\vec N\| =$ area distortion factor $\frac{\Delta\sigma}{\Delta A}$, dir $\vec N = \pm \hat n$ **unit normal vector.**

Surface Integral of scalar: $\displaystyle \iint_S f\, d\sigma := \lim_{\Delta\sigma \to 0} \sum f \cdot \Delta\sigma$ $\approx \|\vec N\| \cdot \Delta A$

Flux: $\displaystyle \iint_S \vec F \cdot \hat n\, d\sigma := \lim_{\Delta\sigma \to 0} \sum (\vec F \cdot \hat n)\, \Delta\sigma$ $\approx \pm \vec N \cdot \Delta A$

Computation: $\boxed{\hat n\, d\sigma = \pm(\vec r_u \times \vec r_v)\, du\, dv}$, $\boxed{d\sigma = \|\vec r_u \times \vec r_v\|\, du\, dv}$

Ex: $\vec{F} = \langle x, y, 0 \rangle$, S sphere of radius a, $\hat{n}$ outward



Parametrization $\vec{r}(\varphi, \theta) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi \rangle$
"spherical coord"

$$\vec{r_\varphi} = \langle a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi \rangle$$
$$\vec{r_\theta} = \langle -a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0 \rangle$$

$$\Rightarrow \vec{r_\varphi} \times \vec{r_\theta} = \langle a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi \rangle$$
$$= a^2\sin\varphi \underbrace{\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle}_{= +\hat{n}}$$

Hence: $\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \int_0^{2\pi}\int_0^{\pi} a^3\sin^3\varphi\,d\varphi\,d\theta = \frac{8\pi}{3}a^3$. $\square$

$\langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, 0 \rangle \cdot \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle \cdot a^2\sin\varphi\,d\varphi\,d\theta$

∇ notation: $\quad \nabla$ "del" $= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

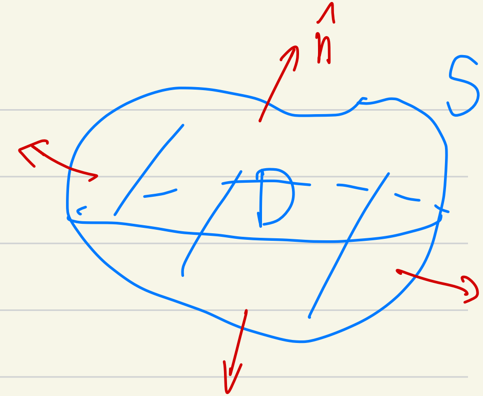
- $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =: \operatorname{grad} f$ gradient

- $\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \boxed{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} =: \operatorname{div} \vec{F}}$ divergence

- $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \boxed{\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle =: \operatorname{curl} \vec{F}}$ Curl

- $\nabla^2 f = \nabla \cdot (\nabla f) = \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} =: \Delta f}$ Laplacian

Divergence Theorem: S closed surface enclosing D oriented with $\hat{n}$ outwards



$\vec{F}$ defined everywhere in D.

Then $$\boxed{\oiint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_D \operatorname{div}\vec{F}\, dV}$$

or, $$\oiint_S \langle P, Q, R\rangle\cdot\hat{n}\, d\sigma = \iiint_D (P_x + Q_y + R_z)\, dV$$

Physical intuition:

the amount of fluid flows out of S — the amount of fluid generated by sources in S.

Ex: $\vec{F} = \langle x, y, 0\rangle$. $\operatorname{div}\vec{F} = 2$. $\Rightarrow \underset{\substack{S\\ \uparrow\\ \text{sphere of radius } a}}{\iint} \vec{F}\cdot\hat{n}\, d\sigma = \underset{\substack{D\\ \uparrow\\ \text{ball of radius } a}}{\iiint} 2\cdot dV = 2\cdot \mathrm{Vol}(D) = \frac{8\pi}{3}a^3$

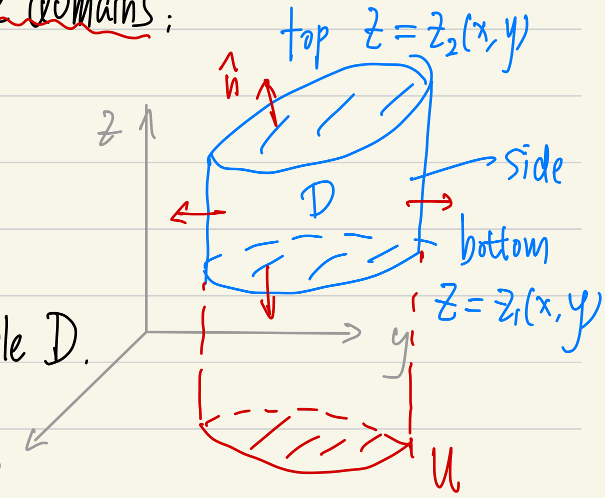
Proof of divergence thm

(1) Simplify integral: Only need to prove $\oiint_S \langle 0, 0, R \rangle \hat{n}\, d\sigma = \iiint_D R_z\, dV$

then get the general case by summing three such identities, one for each cpt.

(2) Simplify domain: Decompose D into vertically simple domains;

Prove the identity for each small pieces, and sum together to get the general case.



(3). Main Part: $\vec{F} = \langle 0, 0, R \rangle$ on vertically simple D.

- $$\iiint_D R_z \, dV = \iint_U \left[\int_{z_1(x,y)}^{z_2(x,y)} R_z \, dz \right] dx\,dy = \iint_U R(x,y,z_2(x,y)) - R(x,y,z_1(x,y)) \, dx\,dy$$

- $$\oiint_S = \iint_{top} + \iint_{bottom} + \iint_{sides}$$

$$\iint_{top} = \iint_U \langle 0, 0, \overset{= R(x,y,z_2(x,y)) \text{ on top}}{R} \rangle \cdot \langle -\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, \overset{\hat{n} \text{ upwards}}{1} \rangle \, dx\,dy = \iint_U R(x,y,z_2(x,y)) \, dx\,dy$$

$$\iint_{bottom} = \iint_U \langle 0, 0, \overset{= R(x,y,z_1(x,y)) \text{ on bottom}}{R} \rangle \cdot \langle \frac{\partial z_2}{\partial x}, \frac{\partial z_2}{\partial y}, \overset{\hat{n} \text{ downwards}}{-1} \rangle \, dx\,dy = \iint_U -R(x,y,z_1(x,y)) \, dx\,dy$$

$\iint_{sides} = 0$ Since $\langle 0, 0, R \rangle$ is normal to $\hat{n}$. $\square$

(S graphs: $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$: $\vec{r}_x = \langle 1, 0, f_x \rangle$ $\Rightarrow$ $\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle$

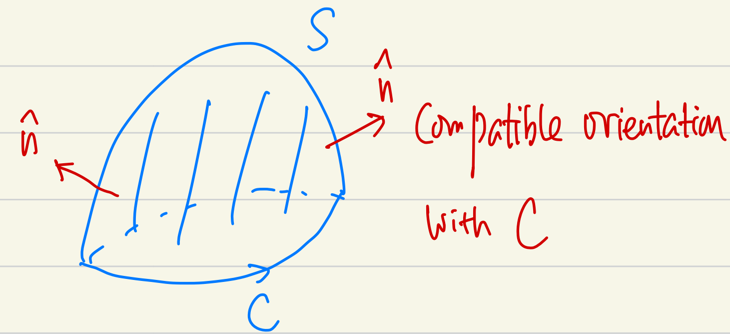
$\vec{r}_y = \langle 0, 1, f_y \rangle$ So $\hat{n}\, d\sigma = \pm \langle -f_x, -f_y, 1 \rangle \, dx\,dy$)

Stokes' Theorem (in $\mathbb{R}^3$)

C : simple closed curves.

S : Any surfaces bounded by C

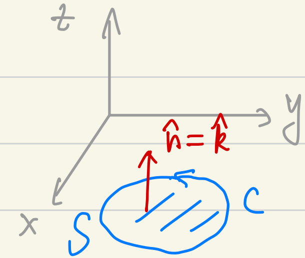
$\vec{F}$: smoothly defined everywhere on S



Then :

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \, d\sigma}$$

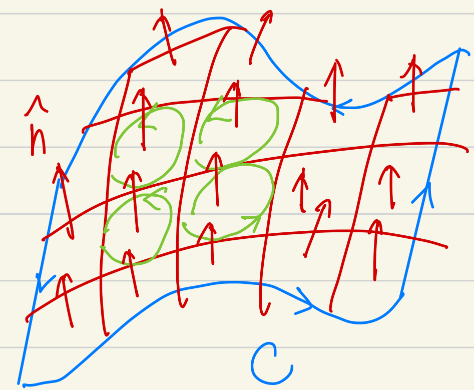
Idea of pf: • True for C, S in xy-plane:



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy \underset{\substack{\uparrow \\ \text{(Green's thm)}}}{=} \iint_S (Q_x - P_y)\,dx\,dy = \iint_S \operatorname{curl}\vec{F}\cdot \underbrace{\hat{n}\,d\sigma}_{\substack{\parallel \\ \hat{k}\,dx\,dy}}$$

• True for C, S in any planes

• Given any S, decompose it into tiny, almost flat pieces.



Sum of flux through each piece = total flux through S

" work around " = total work along C. □

- Stokes and Path-Independent

$(\nabla\times(\nabla f)\equiv 0)$

Recall $\vec{F}$ gradient $\Leftrightarrow$ path-indep $\Leftrightarrow$ Conservative $\Rightarrow$ Curl $\vec{F}=0$

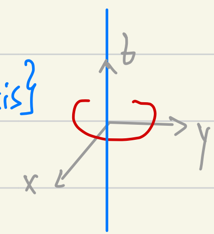
Thm: On a Simply-Connected region $\Omega\subset\mathbb{R}^3$, $\vec{F}$ gradient / path-indep / conservative $\Leftarrow$ Curl $\vec{F}=0$

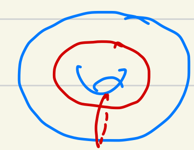
(i.e., every closed loop inside it bound a surface inside it.)

Ex: $\mathbb{R}^3-\{0\}$,

sphere S^2;

$\mathbb{R}^3-\{z\text{-axis}\}$



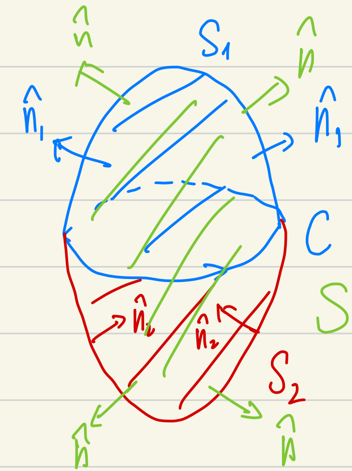


torus T^2 surface

Simply-Connected

NOT Simply-connected

- Stokes and Surface Independence



Suppose S_1, S_2 both bound C. Then Stokes Thm

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot\hat{n}_1\, d\sigma \underset{\text{why same?}}{=} \iint_{S_2} \operatorname{curl}\vec{F}\cdot\hat{n}_2\, d\sigma$$

$$\iint_{S_1} \operatorname{curl}\vec{F}\cdot\hat{n}_1\, d\sigma - \iint_{S_2} \operatorname{curl}\vec{F}\cdot\hat{n}_2\, d\sigma = \oiint_{S=S_1-S_2} \operatorname{curl}\vec{F}\cdot\hat{n}\, d\sigma$$

$$\overset{\text{(Divergence Thm)}}{=} \iiint_D \operatorname{div}(\operatorname{curl}\vec{F})\, dV$$

On the other hand, we know $\operatorname{div}(\operatorname{curl}\vec{F}) = \nabla\cdot(\nabla\times\vec{F}) = \nabla\cdot\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$

$$\equiv 0$$