§ Fubini's Thm:

$$\begin{array}{ccccc} \mathbb{R}^n & = & \mathbb{R}^m & \times & \mathbb{R}^k \\ \cup & & \cup & & \cup \\ R & = & A & \times & B \end{array}$$

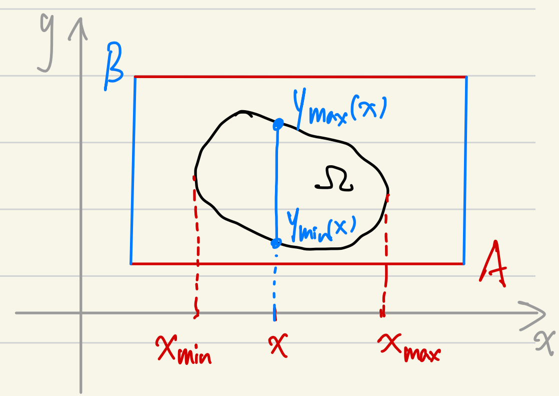
The most useful version: Suppose f is continuous ($\Rightarrow$ integrable) on rectangle $R = A \times B$

Then
$$\boxed{\int_R f\,dV = \int_A \int_B f(x,y)\,dy\,dx = \int_B \int_A f(x,y)\,dx\,dy}$$

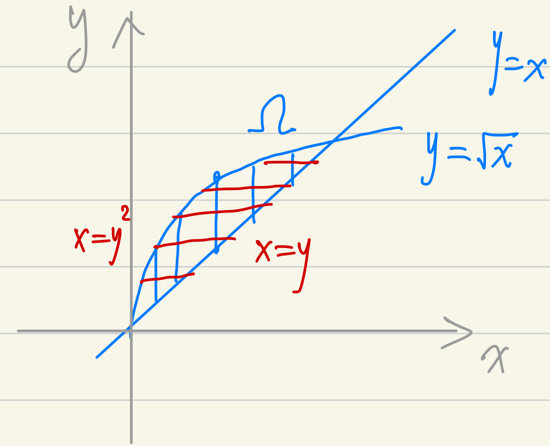
For general integrable function f, need $\overline{\int}$ or $\underline{\int}$. pf standard.

- Arbitrary Ω:
$$\int_\Omega f\,dV = \int_R \bar{f}\,dV = \int_A \int_B \bar{f}(x,y)\,dy\,dx$$

(In 2-d) $$= \int_{x_{min}}^{x_{max}} \int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy\,dx$$



Example 1: $f(x,y) = \frac{e^y}{y}$, Ω bounded by $y = x$ and $y = \sqrt{x}$



then $\int_\Omega f\,dV = \int_0^1 \underbrace{\left(\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy \right)}_{\text{tricky}} dx$

or $= \int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy$

Inner: $\left[\frac{e^y}{y}\cdot x\right]\Big|_{y^2}^{y} = e^y - y\cdot e^y$

Outer: $\int_0^1 (e^y - y\cdot e^y)\,dy = \left[-ye^y + 2e^y\right]\Big|_0^1$
$= e - 2$

$\square$

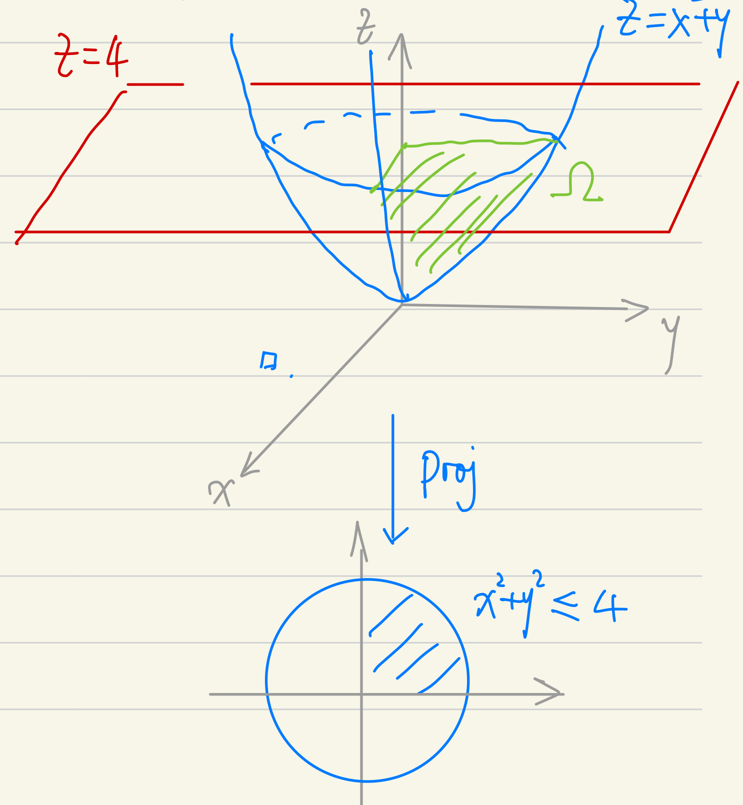
Example 2: Evaluate $\int_{\Omega} x\, dV$ where Ω above $z=x^2+y^2$, below $z=4$, in 1^{st} octant

$$\Omega = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x, y, z \geq 0 \\ x^2+y^2 \leq z \leq 4 \end{array} \right\}$$

- $\displaystyle\int_{\Omega} x\, dV = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} x \; dz \, dy \, dx$

$\cdots$

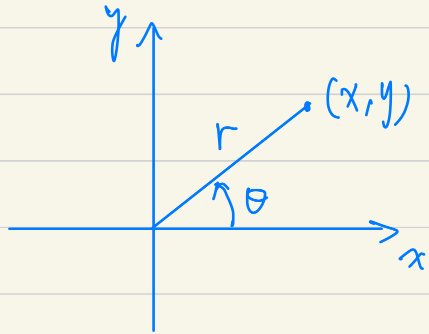
$= \dfrac{64}{15}$



(2-D) polar coordinates.

$\Delta A \approx \Delta r \cdot r \Delta\theta$

So $dA = r\,dr\,d\theta$

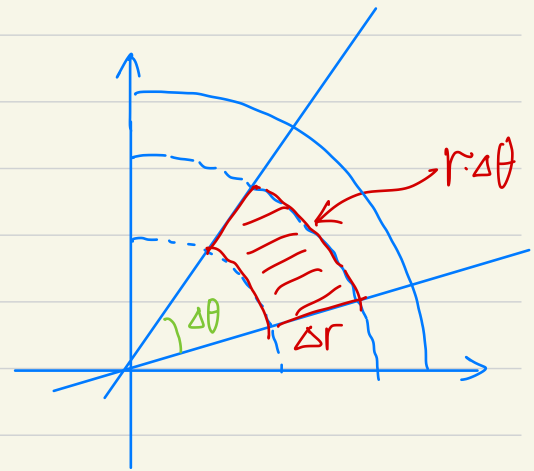


$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

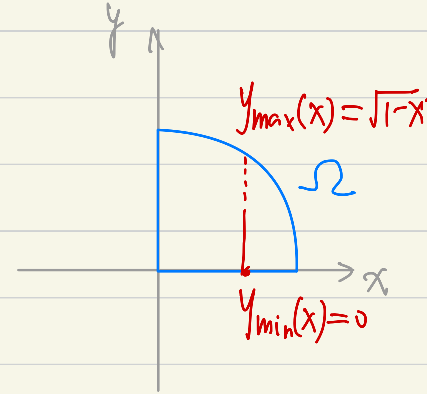
$$\Rightarrow \boxed{\iint_R f\,dA = \int_{\theta_{min}}^{\theta_{max}} \int_{r_{min}(\theta)}^{r_{max}(\theta)} f \cdot r \; dr\, d\theta}$$

don't forget!



Ex: $\displaystyle\iint_{\Omega} (1-x^2-y^2)\,dA$ $\quad \Omega = $ quarter-disk

$\{x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$

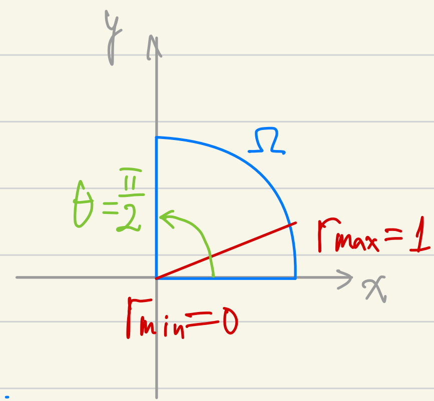


• In Rect. Coord: $\displaystyle\int_0^1 \underbrace{\int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\cdot dy}_{=\ \frac{2}{3}(1-x^2)^{\frac{3}{2}}}\, dx$

$\displaystyle = \int_0^1 \cdot \frac{2}{3}(1-x^2)^{\frac{3}{2}}\, dx = \cdots$

• In Polar Coord: Ω: $0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}$

$f = 1-x^2-y^2 = 1-r^2$

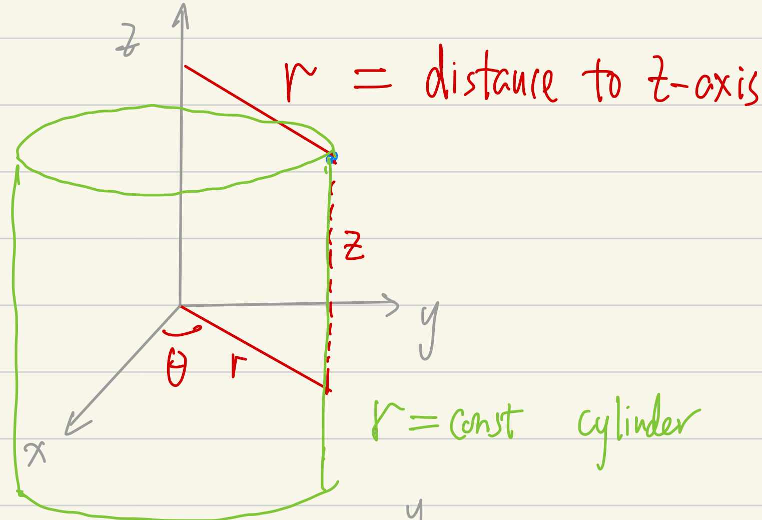


$\displaystyle\int_0^{\frac{\pi}{2}} \underbrace{\int_0^1 (1-r^2)\cdot r\, dr}_{\left[\frac{1}{2}r^2 - \frac{r^4}{4}\right]_0^1 = \frac{1}{4}}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\theta = \frac{\pi}{8}$

(3-D) Cylindrical Coordinate

("rectangle + polar")

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \\ z = z \end{cases}$$

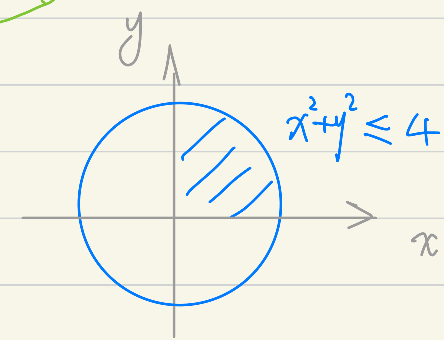


$$\boxed{dV = r\,dr\,d\theta\,dz}$$

Example 2:

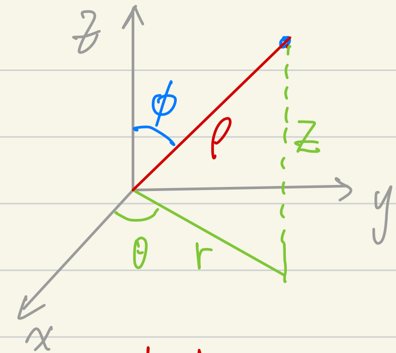
- Using cylindrical coord.

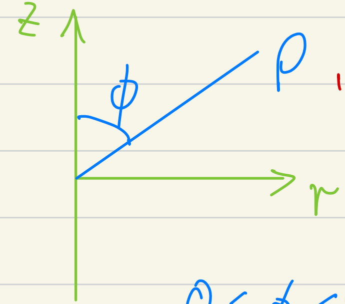
$$\int_0^{\frac{\pi}{2}} \int_0^{2} \int_{r^2}^{4} r \cdot \cos\theta \, dz \, r\,dr\,d\theta = \frac{64}{15}$$



(3-D) Spherical Coordinate

("polar + polar")





$r = \rho \cdot \sin\phi, \quad z = \rho \cdot \cos\phi$

$dr\,dz = \rho\, d\rho\, d\phi$

"half plane"

$0 \le \phi \le \pi$!

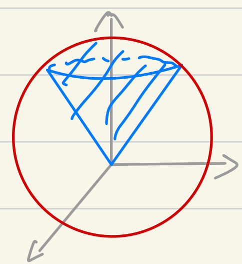
$$\Rightarrow \begin{cases} x = r \cdot \cos\theta = \rho \sin\phi \cdot \cos\theta \\ y = r \cdot \sin\theta = \rho \sin\phi \cdot \sin\theta \\ z = \rho \cdot \cos\phi \end{cases}$$

$$\boxed{dV = r\,dr\,d\theta\,dz = \rho^2 \cdot \sin\phi \cdot d\rho\, d\phi\, d\theta}$$

Ex: "Ice-cream Cone" bounded by

$z = c\sqrt{x^2+y^2}$

and $x^2 + y^2 + z^2 = a^2$



$$\mathrm{Vol}(\Omega) = \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{1}{c})} \int_0^{a} \rho^2 \cdot \sin\phi \, d\rho\, d\phi\, d\theta$$