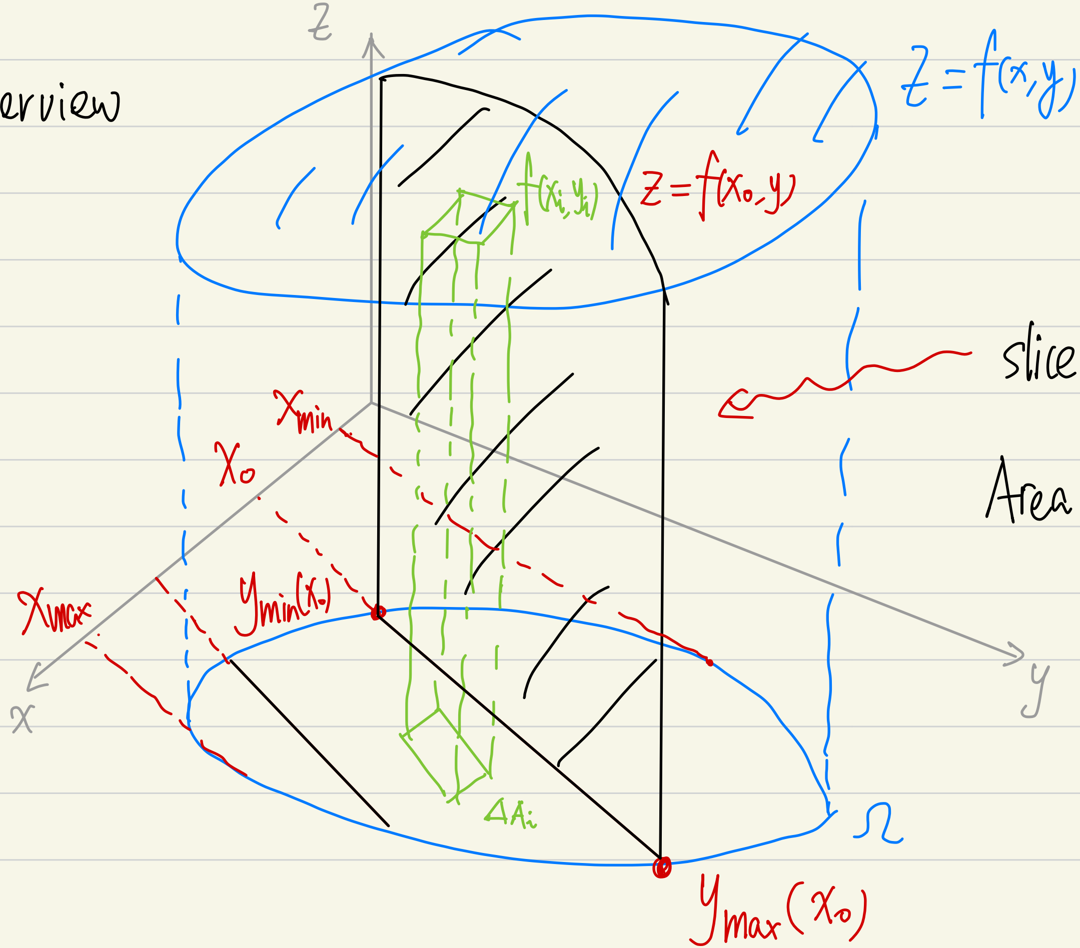
§ Overview



slice by $x = \text{Const.}$

$$\text{Area} = \int_{y_{min}(x_0)}^{y_{max}(x_0)} f(x_0, y)\, dy$$

Double Integral $\iint_\Omega f\,dA$ = Volume below graph $z = f(x,y)$ "Geometrically"

$$= \lim_{\Delta A_i \to 0} \sum_i f(x_i, y_i) \cdot \Delta A_i$$ "Analytically"

$$= \int_{x_{min}}^{x_{max}} S(x)\,dx$$

← area of slice

$$= \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\,dy \right] dx$$ "Computationally"

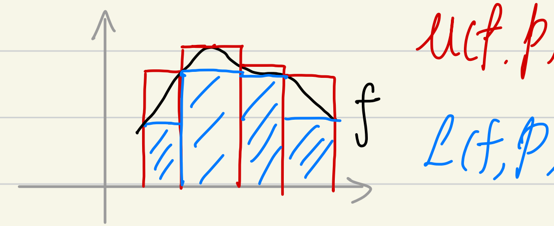
↑ " Fubini's Thm"

← iterated integral.

§ Multi-integral $\int_R f\,dV$ over rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$:

(Assume f bounded)

Partition P:



Upper v.s Lower Sum: $U(f, P) := \sum_{Q \in P} \sup_{x \in Q} f(x) \cdot \mathrm{Vol}(Q) \geq \sum_{Q \in P} \inf_{x \in Q} f(x) \cdot \mathrm{Vol}(Q) =: L(f, P)$

Upper integral $\overline{\int_R} f\,dV := \inf_P U(f, P) \geq \sup_P L(f, P) =: \underline{\int_R} f\,dV$ **lower integral**

- f **integrable** if upper integral = lower integral

$\Longleftrightarrow$ **Riemann Condition**: $\forall \varepsilon, \exists P$ s.t. $U(f, P) - L(f, P) < \varepsilon$.

Main Thm: Continuous function f is integrable on rectangle R

Pf: Note that $U(f,P) - L(f,P) = \sum_Q \left(\sup_{x \in Q} f(x) - \inf_{x \in Q} f(x) \right) \cdot \mathrm{vol}(Q)$ □

by uniform continuity, $< \varepsilon$ if $\mathrm{diam}(Q) < \delta$.

Def: A subset $A \subset \mathbb{R}^n$ has **Content zero** if $\exists$ rectangles $R_1, \dots, R_N$ s.t. $A \subseteq R_1 \cup \dots \cup R_n$ and $\sum_{i=1}^{N} \mathrm{vol}(R_i) < \varepsilon$.

- - - - - - - - **Measure zero** - - - - - - - - - - - - $\{R_i\}_{i=1}^{\infty}$ - - - - and $\sum_{i=1}^{\infty} \mathrm{Vol}(R_i) < \varepsilon$

Thm: f integrable on rectangle R iff f continuous except on a set of measure zero.

Main Idea: $\sum_{Q \in \bigcup_{i=1}^{\infty} R_i} \left(\sup_{x \in Q} f(x) - \inf_{x \in Q} f(x) \right) \cdot \mathrm{vol}(Q)$. ← total volume $< \varepsilon$ by measure zero

$\leq M$ since f bounded

§ Arbitrary bounded subset $\Omega \subset \mathbb{R}^n$.

Define $\boxed{\int_\Omega f\,dV := \int_R \bar{f}\,dV}$ for any rectangle $R \supset \Omega$.

where $\bar{f}(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n - \Omega \end{cases}$ zero-extension of f.

In particular, $\mathrm{Vol}(\Omega) := \int_\Omega 1\,dV$

☆ Thm: A continuous function f is integrable on Ω if $\partial\Omega$ has measure-zero.