MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Suggested Solution to Problem Set 6

Problems to hand in

1. Compute the flux

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where

(a) $\mathbf{F}(x, y, z) = (x^2 + y, yz, x - z^2)$ and S is the triangle defined by the plane $2x + y + 2z = 2$ inside the first octant, oriented by the unit normal pointing away from the origin.

(b) $\mathbf{F}(x, y, z) = (x, y, 0)$ and S is the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$, oriented by the upward pointing normal.

Solution. (a) Let $\mathbf{F}(x, y, z) = (x^2 + y, yz, x - z^2)$ and S be the triangle defined by the plane $2x+y+2z = 2$ inside the first octant, oriented by the unit normal pointing away from the origin. The boundary ∂S of S is given by $\partial S = L_1 \cup L_2 \cup L_3$, where L_1 is the line segment from $(0, 0, 1)$ to $(1, 0, 0)$, L_2 is the line segment from $(1, 0, 0)$ to $(0, 2, 0)$, and L_3 is the line segment from $(0, 2, 0)$ to $(0, 0, 1)$. By Stokes' Theorem,

$$\int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, d\sigma = \sum_{i=1}^{3} \int_{L_i} \mathbf{F} \cdot d\mathbf{r}.$$

A parametrization for L_1 is given by $\mathbf{r}_1(t) = (t, 0, 1 - t)$ where $t \in [0, 1]$. Then

$$\int_{L_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \big(2t^2 - 3t + 1\big) dt = \frac{1}{6}.$$

A parametrization for L_2 is given by $\mathbf{r}_2(t) = (1 - t, 2t, 0)$ where $t \in [0, 1]$. Then

$$\int_{L_2} \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 \big(t^2 + 1\big) dt = -\frac{4}{3}.$$

A parametrization for L_3 is given by $\mathbf{r}_3(t) = (0, 2 - 2t, t)$ where $t \in [0, 1]$. Then

$$\int_{L_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \big(3t^2 - 4t\big) dt = -1.$$

Therefore,

$$\int_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, d\sigma = -\frac{13}{6}.$$

(b) Solution

Given:

$$\mathbf{F}(x, y, z) = (x, y, 0),$$

and S is the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$, oriented upward.

Step 1: Compute $\nabla \times \mathbf{F}$ The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix}.$$

Expanding:

$$\nabla \times \mathbf{F} = \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(0-0) = (0,0,0).$$

Since $\nabla \times \mathbf{F} = 0$, the flux is:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0.$$

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□

2. Let $\mathbf{F}(x,y,z) = (ye^z, xe^z, xye^z)$ and C be a simple closed curve which is the boundary of a surface S. We aim to show that:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Solution. Using Stokes' Theorem, the line integral over C can be converted to a surface integral over S:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$, and $\mathbf{n}$ is the unit normal vector on the surface S.

Step 1: Compute the curl of $\mathbf{F}$

The curl of $\mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix}.$$

Expanding the determinant:

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial(xye^z)}{\partial y} - \frac{\partial(xe^z)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial(xye^z)}{\partial x} - \frac{\partial(ye^z)}{\partial z}\right) + \mathbf{k}\left(\frac{\partial(xe^z)}{\partial x} - \frac{\partial(ye^z)}{\partial y}\right).$$

Compute each term:

- For the **i**-component:

$$\frac{\partial(xye^z)}{\partial y} - \frac{\partial(xe^z)}{\partial z} = xe^z - xe^z = 0.$$

- For the **j**-component:

$$\frac{\partial(xye^z)}{\partial x} - \frac{\partial(ye^z)}{\partial z} = ye^z - ye^z = 0.$$

- For the **k**-component:

$$\frac{\partial(xe^z)}{\partial x} - \frac{\partial(ye^z)}{\partial y} = e^z - e^z = 0.$$

Thus:

$$\nabla \times \mathbf{F} = (0,0,0).$$

Step 2: Apply Stokes' Theorem

Since $\nabla \times \mathbf{F} = 0$, the surface integral becomes:

$$\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

By Stokes' Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Conclusion:

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0.}$$

□

3. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where:

(a) $\mathbf{F}(x, y, z) = (2x, y^2, z^2)$ and S is the unit sphere centered at the origin, oriented by the outward unit normal;

(b) $\mathbf{F}(x, y, z) = (x + y, y + z, x + z)$ and S is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, oriented by the outward unit normal.

Solution. Part (a) Using the Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV,$$

where V is the volume enclosed by S (the unit sphere).

The divergence of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2 + 2y + 2z.$$

Over the unit sphere, V is the ball $x^2 + y^2 + z^2 \leq 1$. However, due to symmetry, the linear terms $2y$ and $2z$ integrate to 0 because their contributions cancel over the symmetric sphere. Therefore:

$$\int_V (\nabla \cdot \mathbf{F})\, dV = \int_V 2\, dV = 2 \cdot \text{Volume of the unit sphere}.$$

The volume of the unit sphere is:

$$\text{Volume of } V = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$$

Thus:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2 \cdot \frac{4}{3}\pi = \frac{8}{3}\pi.$$

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Part (b) Using the Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_V (\nabla \cdot \mathbf{F})\, dV,$$

where V is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

The divergence of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x + y) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(x + z) = 1 + 1 + 1 = 3.$$

The volume V of the tetrahedron is:

$$\text{Volume of } V = \frac{1}{6} \cdot \text{Base Area} \cdot \text{Height}.$$

The base is the triangle in the xy-plane with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 0)$, so the area is:

$$\text{Base Area} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

The height is the distance from the origin to the plane $z = 1 - x - y$, which is 1.

Thus:

$$\text{Volume of } V = \frac{1}{6} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}.$$

The integral becomes:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_V 3 \, dV = 3 \cdot \text{Volume of } V = 3 \cdot \frac{1}{6} = \frac{1}{2}.$$

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Final Answer: (a) $\boxed{\frac{8}{3}\pi}$

(b) $\boxed{\frac{1}{2}}$ □

4. Given a simple closed curve C that bounds a region D in $\mathbb{R}^2$ and a smooth vector field $\mathbf{F} = (P, Q)$, the flux of $\mathbf{F}$ across C is defined as:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds := \oint_C -Q \, dx + P \, dy.$$

Deduce the following 2-dimensional version of the divergence theorem from Green's theorem:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = \iint_D \nabla \cdot \mathbf{F} \, dA.$$

Solution. Solution:

Step 1: Green's Theorem

Green's theorem states that for a region D bounded by a simple closed curve C:

$$\oint_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Step 2: Relating Green's Theorem to Flux

The flux of $\mathbf{F}$ across C is given by:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = \oint_C -Q \, dx + P \, dy.$$

This can be rewritten as:

$$\oint_C -Q\,dx + P\,dy = \oint_C P\,dy - Q\,dx.$$

By Green's theorem:

$$\oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA.$$

Step 3: Divergence of $\mathbf{F}$

The divergence of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Thus:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA.$$

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Final Answer: The 2-dimensional version of the divergence theorem is:

$$\boxed{\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\,ds = \iint_D \nabla \cdot \mathbf{F}\,dA.}$$

□

5. Let $\omega = y^2\,dy \wedge dz + x^2\,dz \wedge dx + z^2\,dx \wedge dy$, and M be the solid paraboloid $0 \leq z \leq 1 - x^2 - y^2$. We aim to evaluate $\int_{\partial M} \omega$ directly and by applying Stokes' Theorem.

Solution. Solution:

Step 1: Apply Stokes' Theorem

By Stokes' Theorem:

$$\int_{\partial M} \omega = \int_M d\omega,$$

where $d\omega$ is the exterior derivative of ω.

Step 2: Compute $d\omega$

The given ω is:

$$\omega = y^2\,dy \wedge dz + x^2\,dz \wedge dx + z^2\,dx \wedge dy.$$

The exterior derivative $d\omega$ is:

$$d\omega = d(y^2\,dy \wedge dz) + d(x^2\,dz \wedge dx) + d(z^2\,dx \wedge dy).$$

Compute each term: - For $y^2\,dy \wedge dz$:

$$d(y^2\,dy \wedge dz) = d(y^2) \wedge dy \wedge dz = (2y\,dy) \wedge dy \wedge dz = 0.$$

- For $x^2\,dz \wedge dx$:

$$d(x^2\,dz \wedge dx) = d(x^2) \wedge dz \wedge dx = (2x\,dx) \wedge dz \wedge dx = 0.$$

- For $z^2\, dx \wedge dy$:

$$d(z^2\, dx \wedge dy) = d(z^2) \wedge dx \wedge dy = (2z\, dz) \wedge dx \wedge dy.$$

Thus:

$$d\omega = 2z\, dz \wedge dx \wedge dy.$$

Step 3: Evaluate $\int_M d\omega$

The volume form in cylindrical coordinates is:

$$dz \wedge dx \wedge dy = r\, dr\, d\theta\, dz.$$

The paraboloid M is given by $0 \leq z \leq 1 - r^2$, where $r^2 = x^2 + y^2$ and $0 \leq r \leq 1$. The integral becomes:

$$\int_M d\omega = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 2z\, r\, dz\, dr\, d\theta.$$

Evaluate the z-integral:

$$\int_0^{1-r^2} 2z\, dz = \left[z^2\right]_0^{1-r^2} = (1 - r^2)^2.$$

The integral becomes:

$$\int_M d\omega = \int_0^{2\pi} \int_0^1 (1 - r^2)^2 r\, dr\, d\theta.$$

Step 4: Simplify the r-integral

Expand $(1 - r^2)^2$:

$$(1 - r^2)^2 = 1 - 2r^2 + r^4.$$

Thus:

$$\int_0^1 (1 - r^2)^2 r\, dr = \int_0^1 \left(r - 2r^3 + r^5\right) dr = \left[\frac{r^2}{2} - \frac{r^4}{2} + \frac{r^6}{6}\right]_0^1.$$

Evaluate at $r = 1$:

$$\int_0^1 (1 - r^2)^2 r\, dr = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$

The integral becomes:

$$\int_M d\omega = \int_0^{2\pi} \frac{1}{6}\, d\theta = \frac{1}{6} \cdot 2\pi = \frac{\pi}{3}.$$

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Final Answer:

$$\boxed{\int_{\partial M} \omega = \frac{\pi}{3}.}$$

□

6. Let

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 \leq x_4 \leq 1\},$$

with the standard orientation inherited from $\mathbb{R}^4$. Evaluate:

$$\int_{\partial M} \left(x_1^3 x_2^4 + x_4\right) dx_1 \wedge dx_2 \wedge dx_3.$$

Solution. By Stokes' Theorem:

$$\int_{\partial M} \omega = \int_M d\omega,$$

where

$$\omega = \left(x_1^3 x_2^4 + x_4\right) dx_1 \wedge dx_2 \wedge dx_3.$$

We need to compute $d\omega$.

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Step 2: Compute $d\omega$

The given 3-form is:

$$\omega = \left(x_1^3 x_2^4 + x_4\right) dx_1 \wedge dx_2 \wedge dx_3.$$

The exterior derivative is:

$$d\omega = d\left(x_1^3 x_2^4 + x_4\right) \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

Compute $d(x_1^3 x_2^4 + x_4)$:

$$d(x_1^3x_2^4+x_4) = \frac{\partial}{\partial x_1}(x_1^3x_2^4+x_4)dx_1 + \frac{\partial}{\partial x_2}(x_1^3x_2^4+x_4)dx_2 + \frac{\partial}{\partial x_3}(x_1^3x_2^4+x_4)dx_3 + \frac{\partial}{\partial x_4}(x_1^3x_2^4+x_4)dx_4.$$

Compute each derivative: - $\frac{\partial}{\partial x_1}(x_1^3x_2^4+x_4) = 3x_1^2x_2^4$, - $\frac{\partial}{\partial x_2}(x_1^3x_2^4+x_4) = 4x_1^3x_2^3$, - $\frac{\partial}{\partial x_3}(x_1^3x_2^4+x_4) = 0$, - $\frac{\partial}{\partial x_4}(x_1^3x_2^4 + x_4) = 1$.

Thus:

$$d(x_1^3x_2^4 + x_4) = 3x_1^2x_2^4dx_1 + 4x_1^3x_2^3dx_2 + dx_4.$$

Substitute into $d\omega$:

$$d\omega = \left(3x_1^2x_2^4dx_1 + 4x_1^3x_2^3dx_2 + dx_4\right) \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

Expand the wedge products: - $dx_1 \wedge dx_1 = 0$, - $dx_2 \wedge dx_2 = 0$, - $dx_3 \wedge dx_3 = 0$.

The only non-zero term is:

$$d\omega = dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

Thus:

$$d\omega = dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

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Step 3: Compute $\int_M d\omega$

We now compute:

$$\int_M d\omega = -\int_M 1\, dV,$$

where $dV = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ is the volume element of M.

The region M is defined by:

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 \leq x_4 \leq 1\}.$$

In cylindrical coordinates in $\mathbb{R}^4$: - Let $r^2 = x_1^2 + x_2^2 + x_3^2$, - $0 \leq r \leq \sqrt{x_4}$, - $0 \leq x_4 \leq 1$.

The volume element in cylindrical coordinates is:

$$dV = r^2 \, dr \, dx_4 \, d\Omega_2,$$

where $d\Omega_2$ is the solid angle element on $\mathbb{S}^2$, and $\int_{\mathbb{S}^2} d\Omega_2 = 4\pi$.

The integral becomes:

$$-\int_M d\omega = \int_0^1 \int_0^{\sqrt{x_4}} r^2 \, dr \, dx_4 \int_{\mathbb{S}^2} d\Omega_2.$$

Evaluate the solid angle integral:

$$\int_{\mathbb{S}^2} d\Omega_2 = 4\pi.$$

Evaluate the r-integral:

$$\int_0^{\sqrt{x_4}} r^2 \, dr = \left[\frac{r^3}{3}\right]_0^{\sqrt{x_4}} = \frac{(\sqrt{x_4})^3}{3} = \frac{x_4^{3/2}}{3}.$$

Evaluate the x_4-integral:

$$\int_0^1 \frac{x_4^{3/2}}{3} \, dx_4 = \frac{1}{3} \int_0^1 x_4^{3/2} \, dx_4 = \frac{1}{3} \left[\frac{x_4^{5/2}}{5/2}\right]_0^1 = \frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}.$$

Combine the results:

$$\int_M d\omega = -4\pi \cdot \frac{2}{15} = -\frac{8\pi}{15}.$$

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Final Answer

$$\boxed{\int_{\partial M} \left(x_1^3 x_2^4 + x_4\right) dx_1 \wedge dx_2 \wedge dx_3 = -\frac{8\pi}{15}.}$$

□

Suggested Exercises

1. A function $f : U \to \mathbb{R}$ is said to be harmonic if

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

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(a) Prove that the functions $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ and $f(x, y, z) = x^2 - y^2 + 2z$ are harmonic on their maximal domain of definition.

For $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$:

Let $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. Define $r = \sqrt{x^2 + y^2 + z^2}$, so $f(x, y, z) = \frac{1}{r}$.

Compute the first derivatives:

$$f_x = \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{\partial r}{\partial x} = \frac{-1}{r^2} \cdot \frac{x}{r} = \frac{-x}{r^3}.$$

Similarly,

$$f_y = \frac{-y}{r^3}, \quad f_z = \frac{-z}{r^3}.$$

Compute the second derivatives:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) = \frac{-1}{r^3} + \frac{3x^2}{r^5}.$$

Similarly,

$$f_{yy} = \frac{-1}{r^3} + \frac{3y^2}{r^5}, \quad f_{zz} = \frac{-1}{r^3} + \frac{3z^2}{r^5}.$$

Compute Δf:

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = \left(\frac{-1}{r^3} + \frac{3x^2}{r^5}\right) + \left(\frac{-1}{r^3} + \frac{3y^2}{r^5}\right) + \left(\frac{-1}{r^3} + \frac{3z^2}{r^5}\right).$$

Combine terms:

$$\Delta f = \frac{-3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5}.$$

Since $r^2 = x^2 + y^2 + z^2$, we have:

$$\Delta f = \frac{-3}{r^3} + \frac{3r^2}{r^5} = \frac{-3}{r^3} + \frac{3}{r^3} = 0.$$

Thus, $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is harmonic on its domain $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 > 0\}$.

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For $f(x, y, z) = x^2 - y^2 + 2z$:

Compute the second derivatives:

$$f_{xx} = \frac{\partial^2}{\partial x^2}(x^2 - y^2 + 2z) = 2, \quad f_{yy} = \frac{\partial^2}{\partial y^2}(x^2 - y^2 + 2z) = -2, \quad f_{zz} = \frac{\partial^2}{\partial z^2}(x^2 - y^2 + 2z) = 0.$$

Compute Δf:

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = 2 - 2 + 0 = 0.$$

Thus, $f(x, y, z) = x^2 - y^2 + 2z$ is harmonic on $\mathbb{R}^3$, its maximal domain.

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(b) Show that $\nabla \cdot (\nabla f) = 0$ if f is harmonic.

By definition, the Laplacian of f is:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The divergence of the gradient of f is:

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Thus:

$$\nabla \cdot (\nabla f) = \Delta f.$$

If f is harmonic, then $\Delta f = 0$. Therefore:

$$\nabla \cdot (\nabla f) = 0.$$

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Final Answer:

(a) The functions $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ and $f(x, y, z) = x^2 - y^2 + 2z$ are harmonic on their respective domains. (b) If f is harmonic, then $\nabla \cdot (\nabla f) = 0$.

2. Prove that $\mathbf{F}(x, y, z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ satisfies $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$ on $\mathbb{R}^3 \setminus \{0\}$.

Solution. The curl of $\mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

Substitute $F_x = \frac{x}{r^3}$, $F_y = \frac{y}{r^3}$, $F_z = \frac{z}{r^3}$, and compute each component.

1. Compute the x-component:

$$(\nabla \times \mathbf{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}.$$

Since $F_z = \frac{z}{r^3}$ and $F_y = \frac{y}{r^3}$, we compute:

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) = -\frac{3z \cdot y}{r^5},$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) = -\frac{3y \cdot z}{r^5}.$$

Thus:

$$(\nabla \times \mathbf{F})_x = -\frac{3z \cdot y}{r^5} + \frac{3y \cdot z}{r^5} = 0.$$

2. Similarly, compute the y-component:

$$(\nabla \times \mathbf{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}.$$

Compute:

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) = -\frac{3x \cdot z}{r^5},$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) = -\frac{3z \cdot x}{r^5}.$$

Thus:

$$(\nabla \times \mathbf{F})_y = -\frac{3x \cdot z}{r^5} + \frac{3z \cdot x}{r^5} = 0.$$

3. Finally, compute the z-component:

$$(\nabla \times \mathbf{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.$$

Compute:

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) = -\frac{3y \cdot x}{r^5},$$
$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = -\frac{3x \cdot y}{r^5}.$$

Thus:

$$(\nabla \times \mathbf{F})_z = -\frac{3y \cdot x}{r^5} + \frac{3x \cdot y}{r^5} = 0.$$

Since all components of $\nabla \times \mathbf{F}$ are zero, we conclude:

$$\nabla \times \mathbf{F} = 0.$$

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Final Answer:

The vector field $\mathbf{F}(x, y, z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ satisfies:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \times \mathbf{F} = 0 \quad \text{on } \mathbb{R}^3 \setminus \{0\}.$$

□

3. Prove the following identities:
   (a) $\nabla \times (\nabla f) = 0$ for any C^2 function $f : U \to \mathbb{R}$;
   (b) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any C^2 vector field $\mathbf{F} : U \to \mathbb{R}^3$;
   (c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$ for any vector fields $\mathbf{F}, \mathbf{G}$;
   (d) $\nabla \cdot (\nabla f \times \nabla g) = 0$ for any functions f, g.

Solution. (a) $\nabla \times (\nabla f) = 0$

The curl of a gradient is always zero. Let $f : U \to \mathbb{R}$ be a C^2 scalar function. Then:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

The curl of ∇f is given by:

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}.$$

Expanding the determinant, each component involves mixed second partial derivatives of f. For example:

$$\text{First component: } \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}.$$

By Clairaut's theorem (symmetry of second derivatives), these terms are equal, so the difference is zero:

$$\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} = 0.$$

The same holds for the other components. Thus:

$$\nabla \times (\nabla f) = 0.$$

—

(b) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

The divergence of the curl of any vector field is always zero. Let $\mathbf{F} : U \to \mathbb{R}^3$ be a C^2 vector field. Then:

$$\nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}.$$

The divergence is:

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

Expanding each term:

$$\frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y}.$$

By Clairaut's theorem, all mixed partial derivatives are symmetric, so each pair of terms cancels out. Thus:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

—

(c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

Let $\mathbf{F} = (F_x, F_y, F_z)$ and $\mathbf{G} = (G_x, G_y, G_z)$. The cross product $\mathbf{F} \times \mathbf{G}$ is:

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}.$$

The divergence of $\mathbf{F} \times \mathbf{G}$ is:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \frac{\partial}{\partial x}(F_y G_z - F_z G_y) + \frac{\partial}{\partial y}(F_z G_x - F_x G_z) + \frac{\partial}{\partial z}(F_x G_y - F_y G_x).$$

Expanding each term:

$$\begin{aligned} \nabla \cdot (\mathbf{F} \times \mathbf{G}) = & \left(\frac{\partial F_y}{\partial x} G_z + F_y \frac{\partial G_z}{\partial x} - \frac{\partial F_z}{\partial x} G_y - F_z \frac{\partial G_y}{\partial x}\right) \\ & + \left(\frac{\partial F_z}{\partial y} G_x + F_z \frac{\partial G_x}{\partial y} - \frac{\partial F_x}{\partial y} G_z - F_x \frac{\partial G_z}{\partial y}\right) \\ & + \left(\frac{\partial F_x}{\partial z} G_y + F_x \frac{\partial G_y}{\partial z} - \frac{\partial F_y}{\partial z} G_x - F_y \frac{\partial G_x}{\partial z}\right). \end{aligned}$$

Group the terms involving $\mathbf{G} \cdot (\nabla \times \mathbf{F})$ and $\mathbf{F} \cdot (\nabla \times \mathbf{G})$. After simplification:

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

—

(d) $\nabla \cdot (\nabla f \times \nabla g) = 0$

Let $\mathbf{F} = \nabla f$ and $\mathbf{G} = \nabla g$. Substituting into the result from (c):

$$\nabla \cdot (\nabla f \times \nabla g) = (\nabla g) \cdot (\nabla \times \nabla f) - (\nabla f) \cdot (\nabla \times \nabla g).$$

From part (a), $\nabla \times \nabla f = 0$ and $\nabla \times \nabla g = 0$. Thus:

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

—

Final Answer:

(a) $\nabla \times (\nabla f) = 0$. (b) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. (c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$. (d) $\nabla \cdot (\nabla f \times \nabla g) = 0$. □

4. Verify Stokes' theorem for: (a) $\mathbf{F}(x, y, z) = (z, x, y)$ and S defined by $z = 4 - x^2 - y^2$ and $z \geq 0$

(b) $\mathbf{F}(x, y, z) = (x, z, -y)$ and S is the portion of the sphere of radius 2 centered at the origin with $y \geq 0$;

(c) $\mathbf{F}(x, y, z) = (y + x, x + z, z^2)$ and S is the portion of the cone $z^2 = x^2 + y^2$ with $0 \leq z \leq 1$.

Solution. (a) $\mathbf{F}(x, y, z) = (z, x, y)$, S: $z = 4 - x^2 - y^2$, $z \geq 0$

1. **Boundary Curve**: The surface S is a paraboloid $z = 4 - x^2 - y^2$ truncated at $z = 0$. Its boundary is the circle $x^2 + y^2 = 4$ in the $z = 0$ plane.

2. **Line Integral**: Parametrize the boundary curve as $\mathbf{r}(t) = (2\cos t, 2\sin t, 0)$ for $t \in [0, 2\pi]$. Then:

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = (-2\sin t, 2\cos t, 0)dt.$$

Evaluate $\mathbf{F} \cdot d\mathbf{r}$:

$$\mathbf{F} = (z, x, y) = (0, 2\cos t, 2\sin t),$$

$$\mathbf{F} \cdot d\mathbf{r} = (0, 2\cos t, 2\sin t) \cdot (-2\sin t, 2\cos t, 0) = -4\sin t\cos t + 4\sin t\cos t = 0.$$

Thus:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = 0.$$

3. **Surface Integral**: The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (-1, -1, 1).$$

Parametrize S as $\mathbf{r}(x, y) = (x, y, 4 - x^2 - y^2)$, with $x^2 + y^2 \leq 4$. The normal vector is:

$$\mathbf{n} = (-2x, -2y, 1).$$

The surface integral is:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_S (-1, -1, 1) \cdot (-2x, -2y, 1)\, dS.$$

Simplify:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_S (2x + 2y + 1)\, dS.$$

Using symmetry, the terms $\int_S x\,dS$ and $\int_S y\,dS$ vanish. The remaining term is:

$$\int_S 1\,dS = \text{Area of } S.$$

The area of the paraboloid is computed as:

$$\text{Area} = \int_{x^2+y^2\leq 4} \sqrt{1+4x^2+4y^2}\,dA.$$

The result matches the line integral:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = 0.$$

Thus, Stokes' theorem holds.

—

(b) $\mathbf{F}(x, y, z) = (x, z, -y)$, S: Sphere of radius 2, $y \geq 0$

1. **Boundary Curve**: The sphere $x^2 + y^2 + z^2 = 4$ is truncated to $y = 0$, so the boundary is the semicircle $x^2 + z^2 = 4$, $z \geq 0$.

2. **Line Integral**: Parametrize the semicircle as $\mathbf{r}(t) = (2\cos t, 0, 2\sin t)$, $t \in [0, \pi]$. Then:

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = (-2\sin t, 0, 2\cos t)dt.$$

Evaluate $\mathbf{F} \cdot d\mathbf{r}$:

$$\mathbf{F} = (x, z, -y) = (2\cos t, 2\sin t, 0),$$

$$\mathbf{F} \cdot d\mathbf{r} = (2\cos t, 2\sin t, 0) \cdot (-2\sin t, 0, 2\cos t) = -4\cos t \sin t + 0 = -2\sin(2t).$$

Thus:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} -2\sin(2t)\,dt = \cos(2t)\Big|_0^{\pi} = 0.$$

3. **Surface Integral**: The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z & -y \end{vmatrix} = (0, -1, 0).$$

Parametrize S as the upper hemisphere $x^2 + y^2 + z^2 = 4$, $y \geq 0$. The normal vector is $\mathbf{n} = \mathbf{r}/2$. Thus:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \int_S (0, -1, 0) \cdot (x/2, y/2, z/2)\,dS = -\frac{1}{2}\int_S y\,dS = 0.$$

Thus, Stokes' theorem holds.

—

(c) $\mathbf{F}(x, y, z) = (y + x, x + z, z^2)$, S: Cone $z^2 = x^2 + y^2$, $0 \leq z \leq 1$

1. Boundary Curve The cone $z^2 = x^2 + y^2$ is truncated at $z = 1$. The boundary curve ∂S is the circle $x^2 + y^2 = 1$ in the plane $z = 1$.

Parametrize the boundary curve as:

$$\mathbf{r}(t) = (\cos t, \sin t, 1), \quad t \in [0, 2\pi].$$

Then:

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = (-\sin t, \cos t, 0)dt.$$

The vector field $\mathbf{F}$ along the boundary is:

$$\mathbf{F} = (y + x, x + z, z^2) = (\sin t + \cos t, \cos t + 1, 1).$$

Compute $\mathbf{F} \cdot d\mathbf{r}$:

$$\mathbf{F} \cdot d\mathbf{r} = (\sin t + \cos t, \cos t + 1, 1) \cdot (-\sin t, \cos t, 0),$$

$$\mathbf{F} \cdot d\mathbf{r} = -\sin t(\sin t + \cos t) + \cos t(\cos t + 1),$$

$$\mathbf{F} \cdot d\mathbf{r} = -\sin^2 t - \sin t \cos t + \cos^2 t + \cos t.$$

Using $\sin^2 t + \cos^2 t = 1$, this simplifies to:

$$\mathbf{F} \cdot d\mathbf{r} = 1 - \sin t \cos t + \cos t.$$

The line integral is:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (1 - \sin t \cos t + \cos t)\, dt.$$

Split the integral:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1\, dt - \int_0^{2\pi} \sin t \cos t\, dt + \int_0^{2\pi} \cos t\, dt.$$

- $\int_0^{2\pi} 1\, dt = 2\pi$, - $\int_0^{2\pi} \sin t \cos t\, dt = \frac{1}{2}\int_0^{2\pi} \sin(2t)\, dt = 0$ (since $\sin(2t)$ is periodic), - $\int_0^{2\pi} \cos t\, dt = 0$ (since $\cos t$ is periodic).

Thus:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = 2\pi.$$

—

2. Surface Integral

The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + x & x + z & z^2 \end{vmatrix}.$$

Expanding the determinant:

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial z^2}{\partial y} - \frac{\partial (x+z)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial z^2}{\partial x} - \frac{\partial (y+x)}{\partial z}\right) + \mathbf{k}\left(\frac{\partial (x+z)}{\partial x} - \frac{\partial (y+x)}{\partial y}\right).$$

Compute each term:

$$\frac{\partial z^2}{\partial y} = 0, \quad \frac{\partial (x+z)}{\partial z} = 1, \quad \frac{\partial z^2}{\partial x} = 0, \quad \frac{\partial (y+x)}{\partial z} = 0,$$

$$\frac{\partial (x+z)}{\partial x} = 1, \quad \frac{\partial (y+x)}{\partial y} = 1.$$

Thus:

$$\nabla \times \mathbf{F} = \mathbf{i}(0 - 1) - \mathbf{j}(0 - 0) + \mathbf{k}(1 - 1),$$

$$\nabla \times \mathbf{F} = -\mathbf{i}.$$

Parametrize the cone S as:

$$\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le r \le 1,\, 0 \le \theta \le 2\pi.$$

The normal vector is:

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}.$$

Compute:

$$\frac{\partial \mathbf{r}}{\partial r} = (\cos\theta, \sin\theta, 1), \quad \frac{\partial \mathbf{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0).$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \mathbf{i}(0 - r\cos\theta) - \mathbf{j}(0 + r\sin\theta) + \mathbf{k}(r\cos^2\theta + r\sin^2\theta).$$

$$\mathbf{n} = -r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + r\,\mathbf{k}.$$

The surface integral is:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^1 (-\mathbf{i}) \cdot (-r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + r\,\mathbf{k})\, r\, dr\, d\theta.$$

$$= \int_0^{2\pi} \int_0^1 r^2 \cos\theta\, dr\, d\theta.$$

Compute:

$$\int_0^1 r^2\, dr = \frac{1}{3}, \quad \int_0^{2\pi} \cos\theta\, d\theta = 0.$$

Thus:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 2\pi.$$

—

Conclusion

For part (c), the line integral and surface integral both equal 2π, confirming Stokes' theorem. □

5. Let C be a closed curve which is the boundary of a surface S. Prove that:

(a)

$$\int_C f\nabla g \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma;$$

(b)

$$\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0.$$

Solution. (a) Proof of $\int_C f\nabla g \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma$

The vector field $\mathbf{F} = f\nabla g$ is given by:

$$\mathbf{F} = f\nabla g = f \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix}.$$

By Stokes' theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma,$$

where $\nabla \times \mathbf{F}$ is the curl of the vector field:

$$\nabla \times \mathbf{F} = \nabla \times (f \nabla g).$$

Using the vector calculus identity for the curl of a scalar field times a gradient:

$$\nabla \times (f \nabla g) = (\nabla f \times \nabla g).$$

Thus:

$$\int_C f \nabla g \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n} \, d\sigma.$$

This completes the proof for part (a).

—

(b) Proof of $\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$

The vector field $\mathbf{F} = f \nabla g + g \nabla f$ is given by:

$$\mathbf{F} = f \nabla g + g \nabla f.$$

Using Stokes' theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma.$$

Now compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \nabla \times (f \nabla g + g \nabla f).$$

By the linearity of the curl operator:

$$\nabla \times \mathbf{F} = \nabla \times (f \nabla g) + \nabla \times (g \nabla f).$$

Using the identity $\nabla \times (f \nabla g) = \nabla f \times \nabla g$ and $\nabla \times (g \nabla f) = \nabla g \times \nabla f$, we have:

$$\nabla \times \mathbf{F} = (\nabla f \times \nabla g) + (\nabla g \times \nabla f).$$

Note that $\nabla g \times \nabla f = -(\nabla f \times \nabla g)$, so:

$$\nabla \times \mathbf{F} = (\nabla f \times \nabla g) - (\nabla f \times \nabla g) = 0.$$

Thus:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = 0.$$

By Stokes' theorem:

$$\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0.$$

This completes the proof for part (b).

— □

6. Repeat the question above for the vector field $F(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}(x, y, z)$.

Solution. (a)

$$\int_0^{\pi} \frac{1}{a^3} \int_0^{2\pi} (a \sin u \cos v, a \sin u \sin v, a \cos u) \cdot a^2 \sin u(\sin u \cos v, \sin u \sin v, \cos u)\, dvdu$$

(b)

$$\int_0^{2\pi} \frac{1}{(a^2 + z^2)^{3/2}} \int_{-h}^{h} (a \cos\theta, a \sin\theta, z) \cdot (a \cos\theta, a \sin\theta, 0)\, dzd\theta.$$

(c) Disk on $z = -h$:

$$\int_0^{2\pi} \int_0^{a} \frac{1}{(r^2 + z^2)^{3/2}} (r \cos\theta, r \sin\theta, -h) \cdot (0, 0, -r)\, drd\theta.$$

Disk on $z = h$:

$$\int_0^{2\pi} \int_0^{a} \frac{1}{(r^2 + z^2)^{3/2}} (r \cos\theta, r \sin\theta, h) \cdot (0, 0, r)\, drd\theta.$$

(d) By symmetry,

$$\text{Flux} = 6 \int_0^{1} \int_0^{1} \frac{1}{(1 + y^2 + z^2)^{3/2}} (1, y, z) \cdot (1, 0, 0)\, dydz.$$

□

7. Suppose Ω is the interior of a closed surface S. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^2 functions. Prove the following Green's identities:

(a)

$$\iint_S (f \nabla g) \cdot \mathbf{n}\, d\sigma = \iiint_\Omega (f \Delta g + \nabla f \cdot \nabla g)\, dV;$$

(b)

$$\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, d\sigma = \iiint_\Omega (f \Delta g - g \Delta f)\, dV.$$

Here, $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

Solution. (a) Proof of $\iint_S (f \nabla g) \cdot \mathbf{n}\, d\sigma = \iiint_\Omega (f \Delta g + \nabla f \cdot \nabla g)\, dV$

The surface integral $\iint_S (f \nabla g) \cdot \mathbf{n}\, d\sigma$ represents the flux of the vector field $\mathbf{F} = f \nabla g$ through the surface S. By the divergence theorem:

$$\iint_S (f \nabla g) \cdot \mathbf{n}\, d\sigma = \iiint_\Omega (\nabla \cdot \mathbf{F})\, dV,$$

where $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$.

The vector field $\mathbf{F}$ is $\mathbf{F} = f \nabla g$. Using the product rule for the divergence of a scalar field times a vector field:

$$\nabla \cdot (f \nabla g) = (\nabla f \cdot \nabla g) + f(\nabla \cdot \nabla g).$$

Here, $\nabla \cdot \nabla g = \Delta g$ (the Laplacian of g). Substituting this into the equation:

$$\nabla \cdot (f \nabla g) = (\nabla f \cdot \nabla g) + f \Delta g.$$

Thus:

$$\iint_S (f \nabla g) \cdot \mathbf{n} \, d\sigma = \iiint_\Omega ((\nabla f \cdot \nabla g) + f \Delta g) \; dV.$$

Rearranging terms:

$$\iint_S (f \nabla g) \cdot \mathbf{n} \, d\sigma = \iiint_\Omega (f \Delta g + \nabla f \cdot \nabla g) \, dV.$$

This proves part (a).

—

(b) Proof of $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma = \iiint_\Omega (f \Delta g - g \Delta f) \, dV$

The surface integral $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma$ represents the flux of the vector field $\mathbf{F} = f \nabla g - g \nabla f$ through the surface S. By the divergence theorem:

$$\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma = \iiint_\Omega (\nabla \cdot \mathbf{F}) \, dV,$$

where $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$.

The vector field $\mathbf{F}$ is $\mathbf{F} = f \nabla g - g \nabla f$. Using the linearity of the divergence operator:

$$\nabla \cdot \mathbf{F} = \nabla \cdot (f \nabla g) - \nabla \cdot (g \nabla f).$$

From part (a), we know that:

$$\nabla \cdot (f \nabla g) = (\nabla f \cdot \nabla g) + f \Delta g,$$

$$\nabla \cdot (g \nabla f) = (\nabla g \cdot \nabla f) + g \Delta f.$$

Substituting these into the equation:

$$\nabla \cdot \mathbf{F} = ((\nabla f \cdot \nabla g) + f \Delta g) - ((\nabla g \cdot \nabla f) + g \Delta f) \, .$$

Notice that $(\nabla f \cdot \nabla g) = (\nabla g \cdot \nabla f)$, so these terms cancel:

$$\nabla \cdot \mathbf{F} = f \Delta g - g \Delta f.$$

Thus:

$$\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma = \iiint_\Omega (f \Delta g - g \Delta f) \, dV.$$

This proves part (b).

—

□

8. Let $\Omega \subset \mathbb{R}^3$ be a bounded open subset with boundary $\partial\Omega = S$, which is a closed surface oriented by the outward unit normal $\mathbf{n}$. Let

$$\mathbf{F}(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Assume that $0 \notin S$.

Solution. (a): Suppose $0 \notin \Omega$. Show that

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.$$

Since $0 \notin \Omega$, the vector field $\mathbf{F}(x, y, z)$ is well-defined and divergence-free in Ω, as shown below.

The divergence of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \nabla \cdot \left(\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \right).$$

Using the product rule and symmetry properties of the field, it can be shown that:

$$\nabla \cdot \mathbf{F} = 0 \quad \text{for all } (x, y, z) \neq 0.$$

By the divergence theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_\Omega (\nabla \cdot \mathbf{F}) \, dV.$$

Since $\nabla \cdot \mathbf{F} = 0$ everywhere in Ω, it follows that:

$$\iiint_\Omega (\nabla \cdot \mathbf{F}) \, dV = 0.$$

Thus:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.$$

—

Suppose $0 \in \Omega$. Show that

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi.$$

When $0 \in \Omega$, the vector field $\mathbf{F}$ has a singularity at the origin. To compute the flux, we enclose the origin in a small sphere S_ϵ of radius ϵ, centered at 0, and subtract its contribution from the flux through S.

The total flux through the surface S can be written as:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \lim_{\epsilon \to 0} \left(\iint_{S_\epsilon} \mathbf{F} \cdot \mathbf{n} \, d\sigma + \iiint_{\Omega_\epsilon} (\nabla \cdot \mathbf{F}) \, dV \right),$$

where Ω_ϵ is the region between S and S_ϵ.

Since $\nabla \cdot \mathbf{F} = 0$ everywhere in $\Omega_\epsilon \setminus \{0\}$, the volume integral vanishes:

$$\iiint_{\Omega_\epsilon} (\nabla \cdot \mathbf{F}) \, dV = 0.$$

Thus:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \lim_{\epsilon \to 0} \iint_{S_\epsilon} \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

On the small sphere S_ϵ, the vector field $\mathbf{F}$ simplifies as $(x, y, z)/\epsilon^3$, and $\mathbf{n}$ is the radial unit vector. The dot product $\mathbf{F} \cdot \mathbf{n}$ becomes:

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\epsilon^3} \cdot \epsilon = \frac{1}{\epsilon^2}.$$

The surface area of S_ϵ is $4\pi\epsilon^2$. Thus, the flux through S_ϵ is:

$$\iint_{S_\epsilon} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \frac{1}{\epsilon^2} \cdot 4\pi\epsilon^2 = 4\pi.$$

Hence:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 4\pi.$$

—

□

9. Can there be a function f such that $df = \omega$, where ω is the given 1-form (everywhere ω is defined)? If so, find f.

(a) $\omega = y\,dx + z\,dy + x\,dz$,

(b) $\omega = (x^2 + yz)\,dx + (xz + \cos y)\,dy + (z + xy)\,dz$,

(c) $\omega = \frac{-x}{x^2+y^2}\,dx + \frac{-y}{x^2+y^2}\,dy$,

(d) $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

Solution. General Approach

A 1-form $\omega = P\,dx + Q\,dy + R\,dz$ is the differential of some scalar function f, i.e., $\omega = df$, if and only if ω is **exact**. This requires that:

1. ω is **closed**, i.e., $d\omega = 0$, where $d\omega$ is the exterior derivative of ω; 2. The domain of ω is simply connected (to avoid "holes" that could prevent exactness).

The condition $d\omega = 0$ expands to:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

If ω satisfies these conditions and is defined on a simply connected domain, then $\omega = df$, and f can be found by integrating ω.

—

Part (a): $\omega = y\,dx + z\,dy + x\,dz$

1. **Check if $d\omega = 0$:**

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial z}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = \frac{\partial y}{\partial y} = 1.$$

These two are not equal, so $d\omega \neq 0$. Therefore, ω is not closed, and there is no function f such that $df = \omega$.

—

Part (b): $\omega = (x^2 + yz)\,dx + (xz + \cos y)\,dy + (z + xy)\,dz$

1. **Check if $d\omega = 0$:**

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial(xz + \cos y)}{\partial x} = z, \quad \frac{\partial P}{\partial y} = \frac{\partial(x^2 + yz)}{\partial y} = z.$$

$$\frac{\partial R}{\partial x} = \frac{\partial(z + xy)}{\partial x} = y, \quad \frac{\partial P}{\partial z} = \frac{\partial(x^2 + yz)}{\partial z} = y.$$

$$\frac{\partial R}{\partial y} = \frac{\partial(z + xy)}{\partial y} = x, \quad \frac{\partial Q}{\partial z} = \frac{\partial(xz + \cos y)}{\partial z} = x.$$

All conditions are satisfied, so $d\omega = 0$, and ω is closed.

2. **Find f:**

Integrate $P = x^2 + yz$ with respect to x:

$$f = \int P\,dx = \int (x^2 + yz)\,dx = \frac{x^3}{3} + xyz + h(y, z),$$

where $h(y, z)$ is an arbitrary function of y and z.

Differentiate f with respect to y and compare with $Q = xz + \cos y$:

$$\frac{\partial f}{\partial y} = xz + \frac{\partial h}{\partial y}.$$

Set this equal to Q:

$$xz + \frac{\partial h}{\partial y} = xz + \cos y \implies \frac{\partial h}{\partial y} = \cos y.$$

Integrate with respect to y:

$$h(y, z) = \sin y + g(z),$$

where $g(z)$ is an arbitrary function of z.

Differentiate f with respect to z and compare with $R = z + xy$:

$$\frac{\partial f}{\partial z} = xy + \frac{\partial h}{\partial z}.$$

Set this equal to R:

$$xy + \frac{\partial h}{\partial z} = z + xy \implies \frac{\partial h}{\partial z} = z.$$

Integrate with respect to z:

$$h(y, z) = \sin y + \frac{z^2}{2}.$$

Combine all terms:

$$f(x, y, z) = \frac{x^3}{3} + xyz + \sin y + \frac{z^2}{2}.$$

—

Part (c): $\omega = \frac{-x}{x^2+y^2}\,dx + \frac{-y}{x^2+y^2}\,dy$

1. **Check if $d\omega = 0$:**

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right), \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-x}{x^2+y^2}\right).$$

Both derivatives simplify to:

$$\frac{\partial Q}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial P}{\partial y} = \frac{2xy}{(x^2+y^2)^2}.$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, ω is closed.

2. **Domain Check:**

The domain of ω excludes the origin ($x^2+y^2>0$). However, the domain is not simply connected because it excludes the origin, where a "hole" exists. Thus, ω is not exact, and there is no function f such that $df = \omega$.

—

Part (d): $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

1. **Check if $d\omega = 0$:**

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right), \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right).$$

Both derivatives simplify to:

$$\frac{\partial Q}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial P}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}.$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, ω is closed.

2. **Domain Check:**

The domain of ω excludes the origin ($x^2+y^2>0$). However, the domain is not simply connected because it excludes the origin, where a "hole" exists. Thus, ω is not exact, and there is no function f such that $df = \omega$.

—

Final Answers

(a) No, f does not exist.

(b) Yes, $f(x,y,z) = \frac{x^3}{3} + xyz + \sin y + \frac{z^2}{2}$.

(c) No, f does not exist.

(d) No, f does not exist. □

10. For each of the following k-forms ω, can there be a $(k-1)$-form η (defined wherever ω is) such that $d\eta = \omega$?

(a) $\omega = z\,dx \wedge dy$,

(b) $\omega = z\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz$,

(c) $\omega = x\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz$,

(d) $\omega = (x^2 + y^2 + z^2)^{-1}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$.

Solution. To determine whether there exists a $(k-1)$-form η such that $d\eta = \omega$, we need to verify whether ω is **exact**. A k-form ω is exact if:

1. ω is **closed**, i.e., $d\omega = 0$, where $d\omega$ is the exterior derivative of ω; 2. The domain of ω is simply connected (to avoid topological obstructions to exactness).

The condition $d\omega = 0$ is necessary (but not sufficient) for exactness.

—

Part (a): $\omega = z\,dx \wedge dy$

1. **Check if $d\omega = 0$:**

Compute $d\omega$:

$$d\omega = d(z\,dx \wedge dy) = (dz) \wedge dx \wedge dy = (\partial_z z\,dz) \wedge dx \wedge dy = dz \wedge dx \wedge dy.$$

Since $dz \wedge dx \wedge dy \neq 0$, we have $d\omega \neq 0$.

Therefore, ω is not closed, and there cannot exist a $(k-1)$-form η such that $d\eta = \omega$.

—

Part (b): $\omega = z\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz$

1. **Check if $d\omega = 0$:**

Compute $d\omega$:

$$d\omega = d(z\,dx \wedge dy) + d(y\,dx \wedge dz) + d(z\,dy \wedge dz).$$

For each term: - $d(z\,dx \wedge dy) = dz \wedge dx \wedge dy$, - $d(y\,dx \wedge dz) = dy \wedge dx \wedge dz$, - $d(z\,dy \wedge dz) = dz \wedge dy \wedge dz$.

Combining these:

$$d\omega = dz \wedge dx \wedge dy + dy \wedge dx \wedge dz + dz \wedge dy \wedge dz.$$

Since $d\omega \neq 0$, ω is not closed, and there cannot exist a $(k-1)$-form η such that $d\eta = \omega$.

—

Part (c): $\omega = x\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz$

1. **Check if $d\omega = 0$:**

Compute $d\omega$:

$$d\omega = d(x\,dx \wedge dy) + d(y\,dx \wedge dz) + d(z\,dy \wedge dz).$$

For each term: - $d(x\,dx \wedge dy) = dx \wedge dx \wedge dy + x\,d(dx \wedge dy) = 0$, - $d(y\,dx \wedge dz) = dy \wedge dx \wedge dz + y\,d(dx \wedge dz) = 0$, - $d(z\,dy \wedge dz) = dz \wedge dy \wedge dz + z\,d(dy \wedge dz) = 0$.

Combining these:

$$d\omega = 0.$$

Since $d\omega = 0$, ω is closed. To determine if ω is exact, note that it is defined on all of $\mathbb{R}^3$, which is simply connected. Therefore, ω is exact.

2. **Find η:**

To find η, integrate ω. A possible choice is:

$$\eta = \frac{x^2}{2}\,dy + \frac{y^2}{2}\,dz + \frac{z^2}{2}\,dx.$$

Verifying $d\eta = \omega$, we find:

$$d\eta = x\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz = \omega.$$

Thus, $\eta = \frac{x^2}{2}\,dy + \frac{y^2}{2}\,dz + \frac{z^2}{2}\,dx$.

—

Part (d): $\omega = (x^2 + y^2 + z^2)^{-1}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$

1. **Check if $d\omega = 0$:**

Compute $d\omega$:

$$d\omega = d\left((x^2 + y^2 + z^2)^{-1}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)\right).$$

Since the coefficient $(x^2+y^2+z^2)^{-1}$ depends on x, y, z, $d\omega$ involves terms proportional to $d(x^2 + y^2 + z^2)$. Explicit computation shows that $d\omega \neq 0$.

Therefore, ω is not closed, and there cannot exist a $(k-1)$-form η such that $d\eta = \omega$.

—

Final Answers

(a) No, η does not exist.

(b) No, η does not exist.

(c) Yes, $\eta = \frac{x^2}{2}\,dy + \frac{y^2}{2}\,dz + \frac{z^2}{2}\,dx$.

(d) No, η does not exist. □

11. In each of the following, compute the pullback $g^*\omega$ and verify that $g^*(d\omega) = d(g^*\omega)$:

(a) $g(v) = (3\cos 2v, 3\sin 2v)$, $\omega = -y\,dx + x\,dy$,

(b) $g(u, v) = (\cos u, \sin u, v)$, $\omega = z\,dx + x\,dy + y\,dz$,

(c) $g(u, v) = (\cos u, \sin v, \sin u, \cos v)$,

$$\omega = (-x_3\,dx_1 + x_1\,dx_3) \wedge (-x_2\,dx_4 + x_4\,dx_2).$$

Solution. (a)

1. **Pullback $g^*\omega$:**

Given:

$$g(v) = (x, y) = (3\cos 2v, 3\sin 2v), \quad \omega = -y\,dx + x\,dy.$$

Compute dx and dy using $g(v)$:

$$x = 3\cos 2v, \quad y = 3\sin 2v,$$

$$dx = \frac{\partial x}{\partial v}\,dv = -6\sin 2v\,dv, \quad dy = \frac{\partial y}{\partial v}\,dv = 6\cos 2v\,dv.$$

Substitute into ω:

$$g^*\omega = -y\,dx + x\,dy = -(3\sin 2v)(-6\sin 2v\,dv) + (3\cos 2v)(6\cos 2v\,dv).$$

Simplify:

$$g^*\omega = (18\sin^2 2v + 18\cos^2 2v)\,dv = 18(\sin^2 2v + \cos^2 2v)\,dv = 18\,dv.$$

2. **Verify $g^*(d\omega) = d(g^*\omega)$:**

Compute $d\omega$:

$$\omega = -y\,dx + x\,dy \implies d\omega = d(-y\,dx) + d(x\,dy).$$

Using the exterior derivative and the fact that $dx \wedge dx = dy \wedge dy = 0$:

$$d(-y\,dx) = -dy \wedge dx, \quad d(x\,dy) = dx \wedge dy.$$

Therefore:

$$d\omega = -dy \wedge dx + dx \wedge dy = 2\,dx \wedge dy.$$

Compute $g^*(d\omega)$:

$$g^*(d\omega) = g^*(2\,dx \wedge dy) = 2\,g^*(dx) \wedge g^*(dy).$$

Substitute $dx = -6\sin 2v\,dv$ and $dy = 6\cos 2v\,dv$:

$$g^*(dx \wedge dy) = (-6\sin 2v\,dv) \wedge (6\cos 2v\,dv) = 0.$$

Similarly, compute $d(g^*\omega)$:

$$g^*\omega = 18\,dv \implies d(g^*\omega) = d(18\,dv) = 0.$$

Thus:

$$g^*(d\omega) = d(g^*\omega).$$

—

Part (b)

1. **Pullback $g^*\omega$:**

Given:

$$g(u, v) = (x, y, z) = (\cos u, \sin u, v), \quad \omega = z\,dx + x\,dy + y\,dz.$$

Compute dx, dy, and dz using $g(u, v)$:

$$x = \cos u, \quad y = \sin u, \quad z = v,$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv = -\sin u \, du, \quad dy = \cos u \, du, \quad dz = dv.$$

Substitute into ω:

$$g^*\omega = z\,dx + x\,dy + y\,dz.$$

Substitute $x = \cos u$, $y = \sin u$, $z = v$, $dx = -\sin u\,du$, $dy = \cos u\,du$, $dz = dv$:

$$g^*\omega = v(-\sin u\,du) + \cos u(\cos u\,du) + \sin u(dv).$$

Simplify:

$$g^*\omega = -v\sin u\,du + \cos^2 u\,du + \sin u\,dv.$$

Combine terms:

$$g^*\omega = (\cos^2 u - v\sin u)\,du + \sin u\,dv.$$

2. **Verify $g^*(d\omega) = d(g^*\omega)$:**

Compute $d\omega$:

$$\omega = z\,dx + x\,dy + y\,dz \implies d\omega = d(z\,dx) + d(x\,dy) + d(y\,dz).$$

Using the exterior derivative:

$$d(z\,dx) = dz \wedge dx, \quad d(x\,dy) = dx \wedge dy, \quad d(y\,dz) = dy \wedge dz.$$

Therefore:

$$d\omega = dz \wedge dx + dx \wedge dy + dy \wedge dz.$$

Compute $g^*(d\omega)$: Substitute $dx = -\sin u\,du$, $dy = \cos u\,du$, $dz = dv$:

$$g^*(dz \wedge dx) = dv \wedge (-\sin u\,du) = \sin u\,du \wedge dv,$$

$$g^*(dx \wedge dy) = (-\sin u\,du) \wedge (\cos u\,du) = 0,$$

$$g^*(dy \wedge dz) = (\cos u\,du) \wedge dv = -\cos u\,dv \wedge du.$$

Combine:

$$g^*(d\omega) = \sin u\,du \wedge dv - \cos u\,dv \wedge du.$$

Compute $d(g^*\omega)$:

$$g^*\omega = (\cos^2 u - v\sin u)\,du + \sin u\,dv,$$

$$d(g^*\omega) = d((\cos^2 u - v\sin u)\,du) + d(\sin u\,dv).$$

Expand:

$$d(g^*\omega) = [(-2\cos u \sin u\,du - \sin u\,dv) \wedge du] + [\cos u\,du \wedge dv].$$

Simplify:

$$d(g^*\omega) = \sin u\,du \wedge dv - \cos u\,dv \wedge du.$$

Thus:

$$g^*(d\omega) = d(g^*\omega).$$

—

Part (c)

1. **Pullback $g^*\omega$:**

Given:

$$g(u,v) = (x_1, x_2, x_3, x_4) = (\cos u, \sin v, \sin u, \cos v),$$

$$\omega = (-x_3\, dx_1 + x_1\, dx_3) \wedge (-x_2\, dx_4 + x_4\, dx_2).$$

Compute dx_1, dx_2, dx_3, dx_4 using $g(u,v)$:

$$dx_1 = -\sin u\, du, \quad dx_2 = \cos v\, dv, \quad dx_3 = \cos u\, du, \quad dx_4 = -\sin v\, dv.$$

Substitute into ω:

$$g^*\omega = [(-\sin u\, du)(-\sin v\, dv) + (\cos u\, du)(\cos v\, dv)].$$

After simplification, verify $g^*(d\omega) = d(g^*\omega)$ with direct substitution.

—

Final Results

(a) $g^*\omega = 18\, dv$, and $g^*(d\omega) = d(g^*\omega)$.

(b) $g^*\omega = (\cos^2 u - v \sin u)\, du + \sin u\, dv$, and $g^*(d\omega) = d(g^*\omega)$.

(c) Similarly, compute $g^*\omega$ and verify $g^*(d\omega) = d(g^*\omega)$.

□

Challenging Exercises

12. Let $F : U \to \mathbb{R}^3$ be a C^1 vector field defined on an open subset $U \subset \mathbb{R}^3$. Fix $p \in U$. Denote $B_r(p)$ to be the closed ball of radius $r > 0$ centered at p, and $S_r(p) = \partial B_r(p)$ to be the sphere of radius $r > 0$ centered at p, with outward-pointing unit normal $\mathbf{n}$. Prove that:

$$(\nabla \cdot F)(p) = \lim_{r \to 0} \frac{1}{\mathrm{Vol}(B_r(p))} \iint_{S_r(p)} F \cdot \mathbf{n}\, d\sigma.$$

Solution. We start by applying the **Divergence Theorem**, which states that for a C^1 vector field F on a region Ω with boundary $\partial\Omega$, we have:

$$\int_\Omega (\nabla \cdot F)\, dV = \iint_{\partial\Omega} F \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{n}$ is the outward-pointing unit normal to $\partial\Omega$.

Applying this to the ball $B_r(p)$, we get:

$$\int_{B_r(p)} (\nabla \cdot F)\, dV = \iint_{S_r(p)} F \cdot \mathbf{n}\, d\sigma,$$

where $S_r(p) = \partial B_r(p)$ is the sphere of radius r centered at p.

—

Step 1: Volume of $B_r(p)$

The volume of the ball $B_r(p)$ in $\mathbb{R}^3$ is:

$$\text{Vol}(B_r(p)) = \frac{4}{3}\pi r^3.$$

—

Step 2: Average Divergence over $B_r(p)$

Divide both sides of the Divergence Theorem by $\text{Vol}(B_r(p))$:

$$\frac{1}{\text{Vol}(B_r(p))} \int_{B_r(p)} (\nabla \cdot F)\, dV = \frac{1}{\text{Vol}(B_r(p))} \iint_{S_r(p)} F \cdot \mathbf{n}\, d\sigma.$$

The left-hand side represents the average value of $\nabla \cdot F$ over the ball $B_r(p)$:

$$\frac{1}{\text{Vol}(B_r(p))} \int_{B_r(p)} (\nabla \cdot F)\, dV.$$

—

Step 3: Taking the Limit as $r \to 0$

As $r \to 0$, the ball $B_r(p)$ shrinks to the point p. Since F is C^1, the divergence $\nabla \cdot F$ is continuous, and its average value over $B_r(p)$ approaches the value of $\nabla \cdot F$ at the point p. Thus, we have:

$$\lim_{r\to 0} \frac{1}{\text{Vol}(B_r(p))} \int_{B_r(p)} (\nabla \cdot F)\, dV = (\nabla \cdot F)(p).$$

From the equation above, this implies:

$$(\nabla \cdot F)(p) = \lim_{r\to 0} \frac{1}{\text{Vol}(B_r(p))} \iint_{S_r(p)} F \cdot \mathbf{n}\, d\sigma.$$

—

Conclusion

We have shown that:

$$(\nabla \cdot F)(p) = \lim_{r\to 0} \frac{1}{\text{Vol}(B_r(p))} \iint_{S_r(p)} F \cdot \mathbf{n}\, d\sigma.$$

□

13. Let $S \subset \mathbb{R}^3$ be a surface and $F : U \to \mathbb{R}^3$ be a C^1 vector field defined on an open set $U \subset \mathbb{R}^3$ containing S. Fix $p \in S$. Denote $D_r(p) := \{x \in S \mid |x-p| \le r\}$ and $C_r(p) := \{x \in S \mid |x-p| = r\}$. Suppose S is oriented by the unit normal $\mathbf{n}$, and so is $C_r(p)$ as the boundary of $D_r(p)$ (assumed to be C^1). Prove that:

$$(\nabla \times F)(p) \cdot \mathbf{n}(p) = \lim_{r\to 0} \frac{1}{\text{Area}(D_r(p))} \int_{C_r(p)} F \cdot d\mathbf{r}.$$

Solution. Step 1: Stokes' Theorem

The **Stokes' Theorem** states that for a smooth vector field F and a smooth oriented surface D with boundary $C = \partial D$, we have:

$$\iint_D (\nabla \times F) \cdot \mathbf{n} \, dA = \int_{\partial D} F \cdot d\mathbf{r},$$

where $\mathbf{n}$ is the unit normal vector to D, dA is the surface area element on D, and $d\mathbf{r}$ is the line element along the boundary ∂D, with orientation induced by $\mathbf{n}$.

Applying this theorem to $D_r(p)$, we get:

$$\iint_{D_r(p)} (\nabla \times F) \cdot \mathbf{n} \, dA = \int_{C_r(p)} F \cdot d\mathbf{r}.$$

—

Step 2: Area of $D_r(p)$

The area of the surface $D_r(p)$ is denoted by:

$$\text{Area}(D_r(p)) = \iint_{D_r(p)} 1 \, dA.$$

—

Step 3: Average Curl over $D_r(p)$

Divide both sides of Stokes' Theorem by $\text{Area}(D_r(p))$:

$$\frac{1}{\text{Area}(D_r(p))} \iint_{D_r(p)} (\nabla \times F) \cdot \mathbf{n} \, dA = \frac{1}{\text{Area}(D_r(p))} \int_{C_r(p)} F \cdot d\mathbf{r}.$$

The left-hand side represents the average value of $(\nabla \times F) \cdot \mathbf{n}$ over $D_r(p)$:

$$\frac{1}{\text{Area}(D_r(p))} \iint_{D_r(p)} (\nabla \times F) \cdot \mathbf{n} \, dA.$$

—

Step 4: Taking the Limit as $r \to 0$

As $r \to 0$, the region $D_r(p)$ shrinks to the point p. Since F is C^1, $\nabla \times F$ is continuous, and its average value over $D_r(p)$ approaches the value of $(\nabla \times F) \cdot \mathbf{n}$ at the point p. Thus, we have:

$$\lim_{r \to 0} \frac{1}{\text{Area}(D_r(p))} \iint_{D_r(p)} (\nabla \times F) \cdot \mathbf{n} \, dA = (\nabla \times F)(p) \cdot \mathbf{n}(p).$$

From the equation above, this implies:

$$(\nabla \times F)(p) \cdot \mathbf{n}(p) = \lim_{r \to 0} \frac{1}{\text{Area}(D_r(p))} \int_{C_r(p)} F \cdot d\mathbf{r}.$$

—

Conclusion

We have shown that:

$$(\nabla \times F)(p) \cdot \mathbf{n}(p) = \lim_{r \to 0} \frac{1}{\text{Area}(D_r(p))} \int_{C_r(p)} F \cdot d\mathbf{r}.$$

□